Solving eikonal equations by the method of characteristics

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1 Introduction

In geometric optics we often need to find local-in-time (that is, short time) solutions to first-order nonlinear equations of the form

$$\phi_t = \lambda(t, x, \phi_x),\ \phi(0, y) = g(y), \tag{1.1}$$

for an unknown function $\phi(t, x)$. Here think of $t \in \mathbb{R}$ as a time variable and $x \in \mathbb{R}^d$ as a space variable, and take $g \in C^\infty$.[1] The real-valued function $\lambda(t, x, \xi)$ is a C^∞ function of $(t, x, \xi) \in \mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_\xi$, which typically arises as an eigenvalue of the matrix symbol $A(t, x, \xi)$ of some matrix differential operator $A(t, x, \partial_x)$. The problem (1.1) is usually referred to as "an eikonal equation" (or eikonal problem).

Let $p = (\tau, \xi) \in \mathbb{R}^{1+d}$ be a placeholder for (ϕ_t, ϕ_x), set $q = (t, x)$, and define the *Hamiltonian*

$$F(q, p) = \tau - \lambda(t, x, \xi). \tag{1.2}$$

The corresponding *Hamiltonian vector field* $H(q, p)$ on $\mathbb{R}^{1+d}_{t,x} \times \mathbb{R}^{1+d}_{\tau,\xi}$ is

$$H(q, p) := \partial_p F \cdot \partial_q - \partial_q F \cdot \partial_p = (\partial_t - \partial_\xi \lambda \cdot \partial_x) - (-\lambda_t \partial_\tau - \lambda_x \cdot \partial_\xi). \tag{1.3}$$

We'll normally write this as $H = (\partial_p F, -\partial_q F) = (F_p, -F_q)$.

Integral curves of H are solutions $(q(s), p(s))$ of the following nonlinear, first-order system of ODEs:

$$(\dot{q}, \dot{p}) = H(p, q) = (F_p(q, p), -F_q(q, p)),\ \ (q, p)(0) = (q_0, p_0), \tag{1.4}$$

where $\dot{q} = \frac{d}{ds} q$. Basic ODE theory implies that solutions exist on *some* time interval $[0, T]$, where $T > 0$. The curves $(q(s), p(s))$ lie in $\mathbb{R}^{1+d}_{t,x} \times \mathbb{R}^{1+d}_{\tau,\xi}$ and are often referred to as *bicharacteristics of* F. The curve $q(s)$ obtained by projecting a bicharacteristic onto the $\mathbb{R}^{1+d}_q$ factor is called a *characteristic*. We will explain how to use bicharacteristics (and characteristics) to solve the eikonal problem (1.1).[2]

Remark 1.1. *Let $(q(s), p(s))$ be a solution of the bicharacteristic equations* (1.4)*. Then* (1.4) *implies*

$$\frac{d}{ds} F(q(s), p(s)) = F_q(q(s), p(s)) \cdot \dot{q}(s) + F_p(q(s), p(s)) \cdot \dot{p}(s) = 0 \text{ for all } s \in [0, T]. \tag{1.5}$$

[1] It will become clear later why it is helpful to use $y \in \mathbb{R}^d$ in (1.1) to denote the space variable on the initial surface $t = 0$.

[2] This method, which has its roots in classical mechanics, is sometimes called the Hamilton-Jacobi method. It is an elementary, but beautiful example of using ODEs to solve PDEs.

Thus, F is constant *and equal to $F(q_0, p_0)$ along the bicharacteristic starting at (q_0, p_0). In classical mechanics, where the Hamiltonian F represents the energy of a physical system whose state at any time is described by position variables q and momentum variables p, a solution of* (1.4) *describes the time evolution of the system.*[3] *In fact, equations* (1.4) *arise by rewriting Newton's second law, force $= ma$, appropriately. The fact that F is constant along bicharacteristics expresses* conservation of energy.[4]

2 Procedure for solving the eikonal equation

In this section we give without justification a procedure for solving the eikonal problem (1.1). In the next section we motivate the steps and provide the justification.

Step 1. For g and λ as in (1.1) solve the bicharacteristic equations (1.4) with initial data

$$q_0(y) = (0, y),\ p_0(y) = (\lambda(0, y, g_y(y)), g_y(y)), \tag{2.1}$$

where y lies in an open set $\Omega \subset\subset \mathbb{R}^d$ containing some designated "basepoint" y_0. This produces functions [5]

$$\begin{aligned} &q(s, y) = (t(s, y), x(s, y)),\ p(s, y) = (\tau(s, y), \xi(s, y)) \text{ such that} \\ &t(0, y) = 0,\ x(0, y) = y,\ \tau(0, y) = \lambda(0, y, g_y(y)),\ \xi(0, y) = g_y(y). \end{aligned} \tag{2.2}$$

Step 2. Now that $q(s, y)$, $p(s, y)$ are determined, define the real-valued function $z(s, y)$ by solving

$$\dot{z}(s, y) = F_p(q(s, y), p(s, y)) \cdot p(s, y),\ z(0, y) = g(y). \tag{2.3}$$

Step 3. Use the inverse function theorem to invert the map $G(s, y) := (t(s, y), x(s, y))$ in a neighborhood of $(0, y_0)$. We compute

$$G'(0, y_0) = \begin{pmatrix} 1 & 0 \\ * & I_{d\times d} \end{pmatrix},$$

so the inverse function theorem applies and yields functions $s = s(t, x)$, $y = y(t, x)$, the components of G^{-1}, which are defined for (t, x) near $(0, y_0)$.

Step 4. Define

$$\begin{aligned} &(a)\ \phi(t, x) = z(s(t, x), y(t, x)) \\ &(b)\ P(t, x) := p(s(t, x), y(t, x)) = (\tau(s(t, x), y(t, x)), \xi(s(t, x), y(t, x))). \end{aligned} \tag{2.4}$$

We claim that

$$P(t, x) = \phi_{t,x}(t, x), \tag{2.5}$$

and that $\phi(t, x)$ is a solution of (1.1) for (t, x) near $(0, y_0)$.[6]

[3] Note from (1.3) and (1.4) that $\dot{t} = 1$, so $t(s, y) = s$ if $t(0, y) = 0$.

[4] In physical problems one can often take the time T of existence of solutions to (1.4) to be ∞.

[5] We now write $q(s, y)$, $p(s, y)$ instead of $q(s)$, $p(s)$ to indicate that the solution depends on the parameter y as well as s.

[6] Here $\phi_{t,x} = \nabla_{t,x}\phi$.

3 Motivation and proof of the method

The procedure given in the last section has a clear geometric meaning that we will now explain.

The interior equation in (1.1) is the statement that $F(t,x,\phi_t(t,x),\phi_x(t,x)) = 0$. To motivate section 2, suppose for a moment that we are *given* some C^2 solution $\phi(t,x)$ of (1.1). Then

$$\phi_t(0,y) = \lambda(0,y,g_y(y)), \phi_x(0,y) = g_y(y),$$

so the functions $q(s,y)$, $p(s,y)$ from step **1** satisfy $(q(0,y),p(0,y)) = (0,y,\phi_t(0,y),\phi_x(0,y))$. Hence $F(q(0,y),p(0,y)) = 0$.

Thus, the initial point of each bicharacteristic curve $(q(s,y),p(s,y))$ lies in the $(2d+1)-$dimensional level set [7]

$$S = \{(q,p) \in \mathbb{R}^{1+d} \times \mathbb{R}^{1+d} : F(q,p) = 0\}, \tag{3.1}$$

and lies on the graph, $\mathcal{G} \subset \mathbb{R}^{1+d} \times \mathbb{R}^{1+d}$ of dimension $1+d$, of the map $(t,x) \to \phi_{t,x}(t,x)$. Remark 1.1 implies

$$F(q(s,y),p(s,y)) = 0 \text{ for all } s \in [0,T], \tag{3.2}$$

so the entire bicharacteristic curve lies in S. Does the entire bicharacteristic curve also lie in $\mathcal{G}$, or in other words is it true that

$$p(s,y) = \phi_{t,x}(q(s,y)) \text{ for all } s \in [0,T]? \tag{3.3}$$

This is possible, since the assumption that ϕ is a solution of (1.1) implies[8]

$$F(q(s,y),\phi_{t,x}(q(s,y))) = 0 \text{ for all } s \in [0,T]. \tag{3.4}$$

To see that (3.3) is true, by uniqueness for ODEs it will suffice to show for some function $\tilde{q}(s,y)$ such that $\tilde{q}(0,y) = (0,y)$, and for $\tilde{p}$ defined by

$$\tilde{p}(s,y) = \phi_{t,x}(\tilde{q}(s,y)) = \phi_q(\tilde{q}(s,y)), \tag{3.5}$$

that $(\tilde{q},\tilde{p})$ satisfy the bicharacteristic equations (1.4).[9] *Define* $\tilde{q}$ to be the solution of [10]

$$\dot{\tilde{q}}(s,y) = F_p(\tilde{q}(s,y),\tilde{p}(s,y)),\ \tilde{q}(0,y) = (0,y),\ \text{where } \tilde{p} \text{ has the form (3.5).} \tag{3.6}$$

Now differentiate (3.5) with respect to s and differentiate $F(\tilde{q},\phi_q(\tilde{q})) = 0$ with respect to $\tilde{q}$ to get

$$\begin{aligned} &\dot{\tilde{p}} = \phi_{qq}\dot{\tilde{q}} \\ &F_q(\tilde{q},\phi_q) + F_p(\tilde{q},\phi_q)\phi_{qq}(\tilde{q}) = 0, \end{aligned} \tag{3.7}$$

and observe that with (3.6) these imply

$$F_q(\tilde{q},\phi_q) + \dot{\tilde{p}} = 0,\ \text{that is } \dot{\tilde{p}} = -F_q(\tilde{q},\tilde{p}). \tag{3.8}$$

By uniqueness for ODEs we conclude that $(q(s,y),p(s,y)) = (\tilde{q}(s,y),\tilde{p}(s,y))$, so (3.3) follows from (3.5); henceforth, we drop the tildes.

[7] The level set S is a C^2 hypersurface in $\mathbb{R}^{1+d} \times \mathbb{R}^{1+d}$.

[8] With step **3** of section 2, (3.4) implies $\mathcal{G} \subset S$ near $(0,y_0,\phi_t(0,y_0),\phi_x(0,y_0))$.

[9] Observe that $\tilde{q}(s,y),\tilde{p}(s,y)$ have the same initial data at $(0,y)$ as q,p. The first component of $\tilde{q}$ is the t component, the second is the $x \in \mathbb{R}^d$ component.

[10] The problem (3.6) is an ODE where the only unknown is $\tilde{q}$.

Remark 3.1. *This shows that the* $(1+d)-$*dimensional graph of* $\phi_{t,x}$*, namely* $\mathcal{G}$*, is the* flowout *under the bicharacteristic flow of the d−dimensional initial submanifold*

$$\mathcal{G}_0 := \{(0, y, \phi_t(0, y), \phi_x(0, y)) : y \in \Omega\} \subset \mathbb{R}^{1+d} \times \mathbb{R}^{1+d}.$$

Thus, the graph of $\phi_{t,x}$ *is* foliated *by the family of disjoint bicharacteristic curves corresponding to distinct points in* $\mathcal{G}_0$*.*

Now *define* $z(s, y) = \phi(q(s, y))$, and compute

$$\dot{z} = \phi_q \dot{q} = p \cdot F_p(q, p), \tag{3.9}$$

where the second equality follows from (1.4) and (3.3). Since $z(0, y) = g(y)$, we have recovered (2.3).

Still working with the assumed solution ϕ of (1.1), we recover equations (2.4)(a) and (2.5) using G^{-1} from step **3**. For example, $\phi(t, x) = z(s(t, x), y(t, x))$ follows from

$$z(s, y) = \phi(q(s, y)) \text{ and } (q \circ G^{-1})(t, x) = (t, x),$$

and (2.5) follows from (3.3).[11] The next proposition summarizes what we have shown so far.

Proposition 3.2. *Suppose* ϕ *is a given* C^2 *solution of the eikonal problem* (1.1) *for* (t, x) *near* $(0, y_0)$*. Then* ϕ must be *the function constructed by the procedure of section 2. In particular, the solution of* (1.1) *is uniquely determined for* (t, x) *near* $(0, y_0)$*.*

Now we drop the assumption that we are *given* a solution of the eikonal problem. We need to show that the function $\phi(t, x)$ constructed in section 2 *is* a solution of the eikonal problem. The uniqueness part of the following theorem has already been proved.

Theorem 3.3. *The function* ϕ *defined by the procedure of section 2 is the unique* C^2 *solution of the eikonal problem* (1.1) *for* (t, x) *near* $(0, y_0)$*.*

Proof. **1.** We use the notation of section 2.[12] From (3.2), which follows just from Remark (1.1) and the fact that $(q(0, y), p(0, y)) \in S$, we obtain using G^{-1} as above that

$$F(t, x, P(t, x)) = 0 \text{ for } P(t, x) \text{ as defined in (2.4).} \tag{3.10}$$

To finish the proof, we therefore only need to show (2.5), that is, $P(t, x) = \phi_{t,x}(t, x)$, where $\phi(t, x)$ is defined in (2.4). We can write this equality as $P(q) = \phi_q(q)$.

2. For $z(s, y)$ as defined in step **2** of section 2, recall that $\phi(t, x) := z(s(t, x), y(t, x))$. In order to compute ϕ_q we therefore need to compute z_s and z_y. We claim [13]

$$\begin{aligned} &(a)\dot{z}(s, y) = p(s, y) \cdot \dot{q}(s, y) \\ &(b)z_y(s, y) = p(s, y) \cdot q_y(s, y). \end{aligned} \tag{3.11}$$

Part (a) is immediate from (2.3) and (1.4). To prove (b) for a given y set

$$r(s) = z_y(s, y) - p(s, y) \cdot q_y(s, y). \tag{3.12}$$

[11] This concludes our motivation of section 2.

[12] This argument mostly follows the proof of Theorem 2 in Evans [E], section 3.2.4.

[13] Equation (3.11)(b) means that $z_{y_i} = p \cdot q_{y_i}$ for $i = 1, \dots, d$.

Using (2.3) and (2.1) we see that

$$r(0) = g_y(y) - p(0,y) \cdot \begin{pmatrix} 0 \\ I_{d\times d} \end{pmatrix} = g_y(y) - g_y(y) = 0.$$

We'll show $\dot{r} = 0$ to conclude $r(s) = 0$ for all s, proving (b).

Differentiate (3.12) with respect to s and (3.11)(a) with respect to y to obtain

$$\begin{aligned} &(a)\ \dot{r} = z_{ys} - (\dot{p} \cdot q_y + p \cdot q_{ys}) \\ &(b)\ z_{sy} = p_y \cdot \dot{q} + \dot{p} \cdot q_{sy}. \end{aligned} \tag{3.13}$$

These imply

$$\dot{r} = p_y \cdot \dot{q} - \dot{p} \cdot q_y = p_y \cdot F_p + F_q \cdot q_y, \tag{3.14}$$

where for the second equality we used (1.4). But differentiating (3.2) with respect to y gives

$$F_q \cdot q_y + F_p \cdot p_y = 0, \text{ so } \dot{r} = 0. \tag{3.15}$$

3. Setting $v = (s,y)$, we can rewrite (3.11) as $z_v(v) = p(v) \cdot q_v(v)$ and use step **3** of section 2 to write $q = q(v(q))$.[14] Using (2.4)(a), which says $\phi(q) = z(v(q))$, and (3.11) we obtain[15]

$$\phi_q(q) = z_v(v(q))v_q(q) = p(v(q)) \cdot q_v(v(q))v_q(q) = p(v(q))q_q(q) = p(v(q)) = P(q). \tag{3.16}$$

□

[14] Observe $v(q) = G^{-1}(q)$ for G as in step **3** of section 2.

[15] Note that $q_q(q) = \frac{\partial(t,x)}{\partial(t,x)}(t,x) = I_{(1+d)\times(1+d)}$.