Hyperbolic boundary problems with large oscillatory coefficients: multiple amplification

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Abstract

We study weakly stable hyperbolic boundary problems with highly oscillatory coefficients that are large, $O(1)$, compared to the small wavelength ϵ of oscillations. Such problems arise, for example, in the study of classical questions concerning the stability of Mach stems and compressible vortex sheets. For such applications one seeks to prove energy estimates that are in an appropriate sense "uniform" with respect to the small wavelength ϵ, but the large oscillatory coefficients are a formidable obstacle to obtaining such estimates. In this paper we analyze a simplified form of the linearized problems that are relevant to the above stability questions, and obtain results that are both positive and negative. On the one hand we identify favorable structural conditions under which it is possible to prove uniform estimates, and then do so by a new approach. We also construct examples showing that large oscillatory coefficients can give rise to an instantaneous *multiple* amplification of the amplitude of solutions relative to data; for example, oscillatory boundary data of a given amplitude $O(1)$ can *immediately* give rise to a solution of amplitude $O(\frac{1}{\epsilon^K})$, where $K \geq 2$.[1] We use the examples of multiple amplification to confirm the optimality of our uniform estimates when the favorable structural conditions hold. When those conditions do not hold, we explain how multiple amplification of infinite order may rule out useful estimates.

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[1]Examples of first-order amplification, where $K = 1$, are well-known [MA88, CG10].

1 Introduction

We study linear hyperbolic boundary problems on $\Omega := \mathbb{R}_t \times \mathbb{R}_{x_1} \times \{x_2 : x_2 \geq 0\}$ of the form

$$\begin{aligned} &L(\partial)u + \mathcal{D}\left(\frac{\phi_{in}}{\epsilon}\right)u := \partial_t u + B_1\partial_{x_1}u + B_2\partial_{x_2}u + \mathcal{D}\left(\frac{\phi_{in}}{\epsilon}\right)u = F\left(t, x, \frac{\phi_0}{\epsilon}\right) \text{ in } x_2 > 0 \\ &Bu = G\left(t, x_1, \frac{\phi_0}{\epsilon}\right) \text{ on } x_2 = 0 \\ &u = 0 \text{ in } t < 0, \end{aligned} \tag{1.1}$$

where the B_j are $N \times N$ constant matrices, B_2 is invertible, and the highly oscillatory matrix coefficient $\mathcal{D}\left(\frac{\phi_{in}}{\epsilon}\right)$ is large, $O(1)$, compared to the small wavelength $\epsilon \in (0, \epsilon_0]$.[2] We take the problem to be weakly stable in the sense that $(L(\partial), B)$ fails to satisfy the uniform Lopatinski condition (Definition 2.5) in a specific way (Assumption 2.9).[3] The boundary matrix B is a constant $p \times N$ matrix of appropriate rank p, and the functions $F(t, x, \theta)$, $G(t, x_1, \theta)$ and $\mathcal{D}(\theta_{in})$ are respectively periodic of period 2π in θ and θ_{in}. The boundary phase is taken to be $\phi_0(t, x_1) = \beta_l \cdot (t, x_1)$, where $\beta_l = (\sigma_l, \eta_l) \in \mathbb{R}^2 \setminus 0$ is one of the "bad" directions where the uniform Lopatinski condition fails. The interior phase

$$\phi_{in}(t, x) = \phi_0(t, x_1) + \omega_N(\beta_l)x_2 \tag{1.2}$$

is one of the incoming phases (Definition 2.6); its restriction to $x_2 = 0$ is ϕ_0. For convenience we take F and G to be zero in $t < 0$.

The problem (1.1) is a simplified form of linearized problems that arise, for example, in the study of stability of Mach stems and vortex sheets. In trying to rigorously justify the classical mechanisms

[2] It is sometimes necessary to replace (F, G) in (1.1) by functions (F^ϵ, G^ϵ) of the same arguments.
[3] Such problems are referred to as "weakly real" (WR) problems in [BGRSZ02].

for Mach stem formation and vortex sheet roll-up proposed in [MR83] and [AM87], one is led to study weakly stable linearized problems with large oscillatory coefficients of the form

$$\begin{aligned}
&\partial_t u+\sum_{i=1}^{2} B_i\left(\epsilon v(t,x,\frac{\phi_{in}}{\epsilon})\right)\partial_{x_i}u+D\left(v(t,x,\frac{\phi_{in}}{\epsilon})\right)u=F\left(t,x,\frac{\phi_0}{\epsilon}\right)\text{ in } x_2>0\\
&B\left(\epsilon v(t,x,\frac{\phi_0}{\epsilon})\right)u=G\left(t,x_1,\frac{\phi_0}{\epsilon}\right)\text{ on } x_2=0\\
&v=0\text{ in } t<0,
\end{aligned}\tag{1.3}$$

where $\epsilon\in(0,\epsilon_0]$ and v, F, G are periodic in their third arguments. The mechanisms in question are exhibited by approximate WKB (or geometric optics) solutions to the original quasilinear equations. One approach to showing that such approximate solutions are close to true exact solutions (i.e., "justifying" the geometric optics solutions) depends essentially on first proving energy estimates for (1.3) that are in an appropriate sense *uniform* with respect to small ϵ (Remark 1.1). Such estimates do not exist at present. Methods that have been used successfully on other hyperbolic boundary problems fail to yield uniform estimates for (1.3).[4] In fact, those methods fail even when applied to the simplified problem (1.1).

We now discuss briefly a few of the new difficulties associated with (1.1). The failure of the uniform Lopatinski (UL) condition results in an energy estimate for $(L(\partial),B):=(\partial_t+B_1\partial_{x_1}+B_2\partial_{x_2},B)$ like

$$|u|_{L^2}\leq C_T\left(|L(\partial)u|_{H^1}+\langle Bu\rangle_{H^1}\right).\tag{1.4}$$

The loss of derivatives in (1.4) means that, when proving estimates for (1.1), we cannot simply treat the term $\mathcal{D}(\frac{\phi_{in}}{\epsilon})u$ in (1.1) as a forcing term, as we could if UL held.[5] It is clear that *even* the smaller term $\epsilon\mathcal{D}(\frac{\phi_{in}}{\epsilon})u$ cannot be treated as a forcing term. In work with Coulombel and Gues [CGW14], we treated this case by an argument that diagonalized $L(\partial)+\epsilon\mathcal{D}(\frac{\phi_{in}}{\epsilon})$ near the bad directions (simultaneous diagonalization).[6] The simultaneous diagonalization argument fails for $L(\partial)+\mathcal{D}(\frac{\phi_{in}}{\epsilon})$; the errors introduced by diagonalization blow up and are not absorbable as $\epsilon\to0$. A new approach is needed.

We emphasize that even when $\mathcal{D}=0$ in (1.1), one observes both a loss of derivatives in the energy estimates and an associated phenomenon of (first-order) *amplification*; data (F,G) of size $O(1)$ generally gives rise to a solution of size $O(\frac{1}{\epsilon})$ [CG10].

It is partly in order to assess the feasibility of proving uniform estimates for problems like (1.3), estimates sufficiently strong to justify geometric optics solutions of the original nonlinear equations, that we study here the problem of proving uniform estimates for (1.1).[7] Also, we regard this question as a natural question in the linear hyperbolic theory worth studying in its own right.

In [Wil20] we studied this question for the problem (1.1) in the special case where $\mathcal{D}(\theta_{in})=e^{i\theta_{in}}I_{N\times N}$. This choice leads to the simplest one-sided cascades (defined below). In this paper we consider general (sufficiently regular) oscillatory $N\times N$ coefficients

$$\mathcal{D}(\theta_{in})=(d_{i,j}(\theta_{in}))_{i,j=1,\dots,N},\tag{1.5}$$

[4] Here we have in mind: (a) problems where the uniform Lopatinski condition holds [Kre70, CGW11]; (b) weakly stable problems like (1.3) but with non-oscillatory coefficients [Cou04, Cou05, CS04]; and (c) weakly stable problems like (1.3) where the oscillatory function v is replaced by ϵv in the arguments of B_i, D, and B [CGW14].

[5] If UL holds, the estimate (1.4) holds with L^2 norms on the right.

[6] More precisely, the simultaneous diagonalization argument in [CGW14] was applied to the *singular system* associated to $L(\partial)+\epsilon\mathcal{D}(\frac{\phi_{in}}{\epsilon})$. Singular systems are defined below in (1.7).

[7] Section 6 gives an assessment.

which produce one-sided cascades when $\mathcal{D}(\theta_{in})$ has only positive Fourier spectrum, and two-sided cascades when $\mathcal{D}(\theta_{in})$ has both positive and negative Fourier spectrum. Here, in addition to proving optimal uniform estimates for certain problems with two-sided cascades, we substantially refine and extend the methods introduced in [Wil20] to obtain estimates that are optimal in the one-sided case. Consequently, the "amplification exponent" $\mathbb{E}$ that appears in the estimates of Theorem 2.11 is generally much smaller than the corresponding exponent in the estimates of [Wil20]. The optimality of $\mathbb{E}$ is verified in section 5 by the construction (and justification) of approximate geometric optics solutions whose amplitudes exhibit exactly the maximum order of multiple amplification relative to the data permitted by the estimate (Remark 5.14).

1.1 Iteration estimate

Here we describe the first main step in our approach to proving uniform estimates for (1.1), which is to prove an *iteration estimate* (1.14) for an associated singular system. The same estimate is used for the cases of one and two-sided cascades. The only restriction on $\mathcal{D}(\theta_{in})$ for the moment is that its Fourier spectrum is contained in $\mathbb{Z} \setminus 0$.[8]

Let us first rewrite (1.1) with slightly modified F as

$$\begin{aligned} &D_{x_2}u + A_0 D_t u + A_1 D_{x_1} u - iB_2^{-1}\mathcal{D}\left(\frac{\phi_{in}}{\epsilon}\right) u = F(t,x,\frac{\phi_0}{\epsilon}) \\ &Bu = G(t,x_1,\frac{\phi_0}{\epsilon}) \text{ on } x_2 = 0 \\ &u = 0 \text{ in } t<0, \end{aligned} \tag{1.6}$$

where $A_0 = B_2^{-1}$, $A_1 = B_2^{-1}B_1$, and $D_{x_i} = \frac{1}{i}\partial_{x_i}$. We study (1.6) by looking for a solution of the form $u(t,x) = U(t,x,\frac{\phi_0}{\epsilon})$, where $U(t,x,\theta)$ is periodic in θ. This yields in the obvious way what we refer to as the corresponding *singular system* for U: [9]

$$\begin{aligned} &D_{x_2}U + A_0(D_t + \frac{\sigma_l}{\epsilon}D_\theta)U + A_1(D_{x_1} + \frac{\eta_l}{\epsilon}D_\theta)U - iB_2^{-1}\mathcal{D}\left(\frac{\omega_N(\beta_l)}{\epsilon}x_2 + \theta\right)U = F(t,x,\theta) \\ &BU = G(t,x_1,\theta) \text{ on } x_2 = 0 \\ &U = 0 \text{ in } t<0. \end{aligned} \tag{1.7}$$

The matrix $\mathcal{D}(\theta_{in})$ can be written

$$\mathcal{D}(\theta_{in}) = \sum_{i,j=1}^N d_{i,j}(\theta_{in})M_{i,j}, \text{ where } d_{i,j}(\theta_{in}) = \sum_{r\in\mathbb{Z}\setminus 0} \alpha_r^{i,j} e^{ir\theta_{in}} \tag{1.8}$$

and $M_{i,j}$ is the $N \times N$ matrix with (i,j) entry equal to 1 and all other entries 0. Using (1.8) we will see (Remark 4.8) that we can reduce the study of (1.7) to the study of the same problem with $\mathcal{D}(\theta_{in})$ replaced by $d(\theta_{in})M$, where M is any constant $N \times N$ matrix and $d(\theta_{in})$ is a scalar periodic function

$$d(\theta_{in}) = \sum_{r\in\mathbb{Z}\setminus 0} \alpha_r e^{ir\theta_{in}}. \tag{1.9}$$

[8]See Remark 2.13(4) for the case where $\mathcal{D}$ has nonzero mean. Also, part (5) of that remark considers what happens when ϕ_{out} is used instead of ϕ_{in} in (1.1).

[9]Clearly, $u = u^\epsilon$ and $U = U^\epsilon$ depend on ϵ, but we usually suppress ϵ-dependence in the notation for these and other functions. Singular systems were used in [JMR95] for initial value problems in free space.

The first equation of (1.7) with $\mathcal{D}(\theta_{in})$ replaced by $d(\theta_{in})M$ can then be written

(1.10)
$$D_{x_2}U + A_0(D_t + \frac{\sigma_l}{\epsilon}D_\theta)U + A_1(D_{x_1} + \frac{\eta_l}{\epsilon}D_\theta)U - i\left(\sum_{r\in\mathbb{Z}\setminus 0}\alpha_r e^{i\left(r\frac{\omega_N(\beta_l)}{\epsilon}x_2 + r\theta\right)}\right)B_2^{-1}MU = F(t,x,\theta).$$

Next we consider the Laplace-Fourier transform in (t, x_1, θ) of the singular system. Expand $U(t,x,\theta) = \sum_{k\in\mathbb{Z}} U_k(t,x)e^{ik\theta}$, expand F and G similarly, set

$$\zeta := (\tau,\eta) := (\sigma - i\gamma, \eta), \text{ where } (\sigma,\eta)\in\mathbb{R}^2,\ \gamma\geq 0, \tag{1.11}$$

and define $V_k(x_2,\zeta) := \widehat{U_k}(\zeta, x_2)$, the Laplace-Fourier transform in (t, x_1) of $U_k(t,x)$. If we define

$$X_k := \zeta + \frac{k\beta_l}{\epsilon} \text{ and } \mathcal{A}(\zeta) = -(A_0\tau + A_1\eta), \tag{1.12}$$

we can write the transformed singular problem for the V_k as :

$$\begin{aligned} &D_{x_2}V_k - \mathcal{A}(X_k)V_k = i\sum_{r\in\mathbb{Z}\setminus 0}\alpha_r e^{ir\frac{\omega_N(\beta_l)}{\epsilon}x_2}B_2^{-1}MV_{k-r} + \widehat{F_k}(x_2,\zeta)\\ &BV_k = \widehat{G_k}(\zeta) \text{ on } x_2 = 0. \end{aligned} \tag{1.13}$$

The *iteration estimate*, proved in Proposition 3.18, is an estimate valid for $\gamma \geq \gamma_0 > 0$ of the form

$$\|V_k\| \leq \frac{C}{\gamma}\sum_{r\in\mathbb{Z}\setminus 0}\sum_{t\in\mathbb{Z}}\|\alpha_r\alpha_t\mathbb{D}(\epsilon,k,k-r)V_{k-r-t}\| + \frac{C}{\gamma^2}\left|\widehat{F_k}|X_k|\right|_{L^2(x_2,\sigma,\eta)} + \frac{C}{\gamma^{3/2}}\left|\widehat{G_k}|X_k|\right|_{L^2(\sigma,\eta)}. \tag{1.14}$$

Here $\|V_k\|$, defined in (3.45), is a modified $L^2(x_2,\sigma,\eta)$ norm, the constants C and γ_0 are independent of (ϵ,ζ,k), and the α_r are as in (1.13) (we redefine α_0 to be 1 in (1.14)).

It is far from clear at this point that an estimate of the form (1.14) is useful. That depends on the behavior of the *global amplification factors* $\mathbb{D}(\epsilon,k,k-r)$, Definition 3.16.[10] One expects these factors, which are functions of ζ, to depend on (ϵ,k,r) and to be *large* sometimes, that is $\gtrsim \frac{1}{\epsilon}$, because of the failure of the uniform Lopatinski condition. In sections 3.3 and 3.5 we show that the $\mathbb{D}(\epsilon,k,k-r)$ can be chosen so that for each ζ:

$$\mathbb{D}(\epsilon,k,k-r)(\zeta) \text{ takes one of three values: } 1, C_5|r|^3, \text{ or } \frac{C_5|r|^3}{\epsilon\gamma}, \tag{1.15}$$

where $C_5\geq 1$ is a fixed positive constant.[11] In particular, the $\mathbb{D}(\epsilon,k,k-r)$ are independent of k! The factors $|r|^3$ in (1.15) are harmless; they are killed by the decay of the α_r. One must understand for a given choice of (ϵ,ζ) how many of these factors can be large, that is, equal to $\frac{C_5|r|^3}{\epsilon\gamma}$; we say more about this below.

[10] Of course, this also depends on having the coefficients α_r decay sufficiently rapidly with respect to r. That is the case provided $d(\theta_{in})$ is sufficiently regular, which we always assume.

[11] As ζ varies, the value taken by $\mathbb{D}(\epsilon,k,k-r)(\zeta)$ can vary.

The norm $\|V_k\|$ satisfies

$$|(\|V_k\|)|_{\ell^2(k)} \gtrsim |e^{-\gamma t}U|_{L^2(t,x,\theta)} + \left|\frac{e^{-\gamma t}U(0)}{\sqrt{\gamma}}\right|_{L^2(t,x_1,\theta)}, \tag{1.16}$$

for $U(t,x,\theta)$ as in (1.7). The second main step in the analysis is to use the iteration estimate to estimate $|(\|V_k\|)|_{\ell^2(k)}$ in terms of the data (F,G), and this leads us to examine the cascades produced by iterating (1.14).

1.2 One and two-sided cascades

We first discuss the one-sided cascades that arise when the coefficients α_r in (1.13) vanish for $r \leq 0$, that is, when $d(\theta_{in})$ has only positive Fourier spectrum.[12] Let us assume for now that $F = 0$ and $G = \sum_{k=N^*}^{\infty} G_k(t,x_1)e^{ik\theta}$ for some $N^* \in \mathbb{Z}$; in fact, take $N^* = 1$ for the moment. Since $G_k = 0$ for $k < 1$, it follows from (1.13) that $V_k = 0$ for $k < 1$, so the iteration estimate (1.14) now reduces to

$$\|V_k\| \leq \frac{C}{\gamma}\sum_{r=1}^{k-1}\sum_{t=0}^{k-r-1} \|\alpha_r\alpha_t\mathbb{D}(\epsilon,k,k-r)V_{k-r-t}\| + |\mathcal{G}_k|_{L^2(\sigma,\eta)},\ k \geq 1, \tag{1.17}$$

where $\mathcal{G}_k = \frac{C}{\sqrt{\gamma}}\frac{\widehat{G_k}|X_k|}{\gamma}$. For any $k \geq 1$ the indices $k-r-t$ that appear in nonzero terms on the right side of (1.17) satisfy $k-r-t < k$.

We can estimate the V_k for $k \geq 1$ in terms of the G_j, $1 \leq j \leq k$ by iterating the estimate (1.17). The iterations quickly become tedious to write down in detail, but let us do the first few. Writing $|\mathcal{G}_j|_{L^2(\sigma,\eta)}$ simply as $|\mathcal{G}_j|$ we obtain:[13]

(1.18)

$$\begin{aligned}
&\|V_1\| \leq |\mathcal{G}_1|\\
&\|V_2\| \leq \frac{C}{\gamma}\|\alpha_1\mathbb{D}(\epsilon,2,1)V_1\| + |\mathcal{G}_2| \leq \frac{C}{\gamma}|\alpha_1\mathbb{D}(\epsilon,2,1)\mathcal{G}_1| + |\mathcal{G}_2|\\
&\|V_3\| \leq \frac{C}{\gamma}\|\alpha_1\mathbb{D}(\epsilon,3,2)V_2\| + \frac{C}{\gamma}\|\alpha_1\mathbb{D}(\epsilon,3,2)V_1\| + \frac{C}{\gamma}\|\alpha_2\mathbb{D}(\epsilon,3,1)V_1\| + |\mathcal{G}_3| \leq\\
&\frac{C}{\gamma}\left[\frac{C}{\gamma}|\alpha_1\mathbb{D}(\epsilon,3,2)\alpha_1\mathbb{D}(\epsilon,2,1)\mathcal{G}_1| + |\alpha_1\mathbb{D}(\epsilon,3,2)\mathcal{G}_2|\right] + \frac{C}{\gamma}|\alpha_1\mathbb{D}(\epsilon,3,2)\mathcal{G}_1| + \frac{C}{\gamma}|\alpha_2\mathbb{D}(\epsilon,3,1)\mathcal{G}_1| + |\mathcal{G}_3|.
\end{aligned}$$

It turns out that all the essential information about the estimate of V_k is contained in the cascade that arises in the obvious way by iteration of (1.17). For example, the cascade corresponding to the estimate of V_3 in (1.18) is the following four-stage cascade:

$$[(V_3)] \to [(V_2,V_1,V_1,\mathcal{G}_3)] \to [(V_1,\mathcal{G}_2)(\mathcal{G}_1)(\mathcal{G}_1)\mathcal{G}_3] \to [(\mathcal{G}_1),\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_1,\mathcal{G}_3]. \tag{1.19}$$

Here brackets $[\cdot]$ mark the individual stages, and a V_j or $\mathcal{G}_j$ term is included in parentheses only when it makes its very first appearance in the cascade.

[12] There is an exactly parallel theory when $d(\theta_{in})$ has only negative Fourier spectrum.

[13] When iterating the estimate (1.17), we repeatedly make use of the fact that functions of ζ like $\mathbb{D}(\epsilon,p,p-r)$ commute right through the problem (1.13).

When $k=5$ the following six-stage cascade can easily be written down *without* first doing a detailed estimate of V_5 as in (1.18):

$$\begin{aligned}
&[(V_5)] \to [(V_4, V_3, V_2, V_1, V_3, V_2, V_1, V_2, V_1, V_1, \mathcal{G}_5)] \to \\
&[(V_3, V_2, V_1, V_2, V_1, V_1, \mathcal{G}_4), (V_2, V_1, V_1, \mathcal{G}_3), (V_1, \mathcal{G}_2), (\mathcal{G}_1), (V_2, V_1, V_1, \mathcal{G}_3), \\
&\quad (V_1, \mathcal{G}_2), (\mathcal{G}_1), (V_1, \mathcal{G}_2), (\mathcal{G}_1), (\mathcal{G}_1), \mathcal{G}_5] \to \\
&[(V_2, V_1, V_1, \mathcal{G}_3), (V_1, \mathcal{G}_2), (\mathcal{G}_1), (V_1, \mathcal{G}_2), (\mathcal{G}_1), (\mathcal{G}_1), \mathcal{G}_4, (V_1, \mathcal{G}_2), (\mathcal{G}_1), \\
&\quad (\mathcal{G}_1), \mathcal{G}_3, (\mathcal{G}_1), \mathcal{G}_2, (V_1, \mathcal{G}_2), (\mathcal{G}_1), (\mathcal{G}_1), \mathcal{G}_3, (\mathcal{G}_1), \mathcal{G}_2, \mathcal{G}_1, (\mathcal{G}_1), \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_5] \to \\
&[(V_1, \mathcal{G}_2), (\mathcal{G}_1), (\mathcal{G}_1), \mathcal{G}_3, (\mathcal{G}_1), \mathcal{G}_2, \mathcal{G}_1, (\mathcal{G}_1), \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_4, (\mathcal{G}_1), \mathcal{G}_2, \mathcal{G}_1, \\
&\quad \mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_1, \mathcal{G}_2, (\mathcal{G}_1), \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_5] \to \\
&[(\mathcal{G}_1), \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_4, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \\
&\quad \mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_5]
\end{aligned} \tag{1.20}$$

In (1.18) and (1.20) the indices that appear on the V_j in the second and subsequent stages are less than the index that appears in the first stage; this defines one-sided cascades.

In section 4 we provide a more efficient way of constructing cascades ($\mathcal{G}_j$-cascades, (4.1)) and show how to reconstruct the estimate of $\|V_k\|$ in terms of the $|\mathcal{G}_j|$ from the cascade. As in (1.18), in the estimate of $\|V_k\|$ terms will occur on the right side in which products of up to $k-1$ global amplification factors appear:

$$\mathbb{D}(\epsilon, k, k_1)\mathbb{D}(\epsilon, k_2, k_3)\mathbb{D}(\epsilon, k_4, k_5)\ldots \text{ where } k > k_1 \geq k_2 > k_3 \geq k_4 > k_5 \ldots. \tag{1.21}$$

It appears that one can obtain a useful final energy estimate of $|(\|V_k\|)|_{\ell^2(k)}$ only in problems where one can show that for any given (ϵ, ζ), the number of large (that is $\frac{C_5|r|^3}{\epsilon\gamma}$) factors in products like (1.21) is bounded above by a number $\mathbb{E}$ that is *independent* of (ϵ, ζ, k). In our first main result, Theorem 2.11, we identify classes of problems for which such an $\mathbb{E}$ exists. For these problems simple formulas for $\mathbb{E}$ (2.27),(2.30) are given in terms of the numbers of incoming and outgoing modes $|\mathcal{O}|$, $|\mathcal{I}|$ and the number of directions $\beta_j \in \Upsilon^0$ where the uniform Lopatinski condition fails.[14] An efficient procedure for estimating $|(\|V_k\|)|_{\ell^2(k)}$ on the basis of estimates like (1.18) is given in the proof of Proposition 4.5. This leads to the following estimate for $U(t, x, \theta)$ as in (1.7): there exist $\epsilon_0 > 0$ and positive constants $\mathbb{E} \in \mathbb{N}$, K, γ_0 independent of ϵ such that for $0 < \epsilon < \epsilon_0$ and $\gamma \geq \gamma_0$ we have[15]

$$\begin{aligned}
&|e^{-\gamma t}U|_{L^2(t,x,\theta)} + \left|\frac{e^{-\gamma t}U(0)}{\sqrt{\gamma}}\right|_{L^2(t,x_1,\theta)} \leq \\
&\quad \frac{K}{(\epsilon\gamma)^{\mathbb{E}}}\left[\frac{1}{\gamma^2}\left(\sum_{k\in\mathbb{Z}}\left||X_k|\widehat{F_k}\right|^2_{L^2(x_2,\sigma,\eta)}\right)^{1/2} + \frac{1}{\gamma^{3/2}}\left(\sum_{k\in\mathbb{Z}}\left|\widehat{G_k}|X_k|\right|^2_{L^2(\sigma,\eta)}\right)^{1/2}\right].
\end{aligned} \tag{1.22}$$

We call the process of passing from the iteration estimate to an estimate like (1.22) *the cascade estimates.*

[14] Here Υ_0 is the set of directions where the uniform Lopatinski condition fails. In 2D problems like (1.1), Υ_0 has the form $\{\pm\beta_1, \ldots, \pm\beta_L\}$ for some unit vectors $\beta_j \in \mathbb{R}^2$ (Proposition 3.1). The sets $\mathcal{I}$, $\mathcal{O}$ are the index sets for the incoming and outgoing modes (Definition 2.6).

[15] The argument with $F = 0$ goes through unchanged if $G = \sum_{k\geq N^*} G_k e^{ik\theta}$ instead of $G = \sum_{k\geq 1} G_k e^{ik\theta}$. The case $G = 0$, $F = \sum_{k\geq N^*} F_k e^{ik\theta}$ can be treated in the same way. The constants K, γ_0, ϵ_0 in (1.22) are independent of N^*, so one can take a limit as $N^* \to -\infty$ to handle general data (F, G).

Remark 1.1. *(1) We refer to an estimate like* (1.22) *as a* uniform *estimate because the constants* $\mathbb{E}$, K, *and* γ_0 *are independent of* ϵ. *In section 5 we use such estimates to justify high order geometric optics expansions for some weakly stable systems like* (1.1). *If, for example, the estimate held only for* $\gamma_0 \gtrsim \frac{1}{\epsilon}$, *it would be useless for this purpose (Remark 5.14, part (3)). Such a condition on* γ_0 *is needed for estimates obtained by applying the approach of [Cou04, Cou05] or [CGW14] to problems with large oscillatory coefficients like* (1.1).

(2) When the uniform Lopatinski condition (Definition 2.5) holds, the estimate (1.22) *holds with* $\mathbb{E} = 0$ *and without the large* $|X_k|$ *factors on the right.*

1.2.1 Two-sided cascades.

Two-sided cascades appear whenever $d(\theta_{in})$ as in (1.9) has both positive and negative Fourier spectrum. First we consider the simplest choice of $d(\theta_{in})$ that gives rise to a two-sided cascade:

$$d(\theta_{in}) = e^{i\theta_{in}} + e^{-i\theta_{in}}. \tag{1.23}$$

With $F = 0$ for now the transformed singular problem (1.13) reduces to

$$\begin{aligned} &D_{x_2}V_k - \mathcal{A}(X_k)V_k = ie^{i\frac{\omega_N(\beta_l)}{\epsilon}x_2}B_2^{-1}MV_{k-1} + ie^{-i\frac{\omega_N(\beta_l)}{\epsilon}x_2}B_2^{-1}MV_{k+1} \\ &BV_k = \widehat{G_k}(\zeta) \text{ on } x_2 = 0. \end{aligned} \tag{1.24}$$

and the iteration estimate (1.14) reduces to

$$\|V_k\| \leq \frac{C}{\gamma} \sum_{r\in\{1,-1\}} \sum_{t\in\{0,1,-1\}} \|\mathbb{D}(\epsilon, k, k-r)V_{k-r-t}\| + |\mathcal{G}_k|_{L^2(\sigma,\eta)},\ k \in \mathbb{Z}, \tag{1.25}$$

Suppose we take $G(t, x_1, \theta) = G_0(t, x_1)$ and attempt to estimate $\|V_0\|$ in terms of $|\mathcal{G}_0|$ by iterations of (1.25) parallel to (1.19), (1.20). We now obtain an *infinite* cascade in which terms V_j for *all* $j \in \mathbb{Z}$ (a two-sided cascade) eventually appear on the right:[16]

(1.26)
$$\begin{aligned} &[(V_0)] \to [(V_{-2}, V_{-1}, V_0, V_0, V_1, V_2, \mathcal{G}_0)] \to [(V_{-4}, V_{-3}, V_{-2}, V_{-2}, V_{-1}, V_0), (V_{-3}, V_{-2}, V_{-1}, V_{-1}, V_0, V_1), \\ &(V_{-2}, V_{-1}, V_0, V_0, V_1, V_2, \mathcal{G}_0), (V_{-2}, V_{-1}, V_0, V_0, V_1, V_2, \mathcal{G}_0), (V_{-1}, V_0, V_1, V_1, V_2, V_3), (V_0, V_1, V_2, V_2, V_3, V_4), \mathcal{G}_0] \\ &\to \dots \end{aligned}$$

From (1.26) we see that the cascade will never terminate in a stage where only entries $\mathcal{G}_0$ appear. Moreover, it is not hard to check that if one writes out the iterated estimates corresponding to (1.26), the following situation will obtain: for any particular (ϵ, ζ, p) and for *any* $\mathbb{E} \in \mathbb{N}$, after enough iterations terms will eventually appear on the right hand side of the estimate in which products of $\mathbb{E}$ copies of $\mathbb{D}(\epsilon, p, p-1)(\zeta)$ occur.[17] If $\mathbb{D}(\epsilon, p, p-1)(\zeta) = \frac{C_5}{\epsilon\gamma}$ ($|r| = 1$ now), then factors $\left(\frac{1}{\epsilon\gamma}\right)^{\mathbb{E}}$ will occur for all $\mathbb{E} \in \mathbb{N}$. Consequently, we are able to obtain a sensible estimate for this problem only when all factors $\mathbb{D}(\epsilon, p, p\pm 1)$, $p \in \mathbb{Z}$ are $\leq C_5$ for all ζ (or almost every ζ). In that case one can estimate $|(\|V_k\|)|_{\ell^2(k)}$ by

[16] In the step $[(V_0)] \to [(V_{-2}, V_{-1}, V_0, V_0, V_1, V_2, \mathcal{G}_0)]$ the terms V_{-1},V_1 correspond to the terms in $(1.25)_{k=0}$ where $r = +1, -1$ respectively, and $t = 0$.

[17] The same applies to $\mathbb{D}(\epsilon, p, p+1)$ of course.

summing (the square of) (1.25) over k and choosing γ large enough to absorb the $\mathbb{D}$ terms on the left. This argument does *not* involve iteration of (1.25), and thus *avoids* estimation of two-sided cascades.

Our second main result, Theorem 2.12, identifies a class of problems like (1.1) with oscillatory coefficients having both positive and negative spectrum and for which the global amplification factors satisfy

$$\mathbb{D}(\epsilon, k, k-r) \leq C_5|r|^3 \text{ for all } (\epsilon, k, r, \zeta). \tag{1.27}$$

The property (1.27) allows us to prove an estimate like (1.22) with $\mathbb{E} = 0$. The proof shows that (1.27) holds whenever

$$\Upsilon_0 = \{\beta_l, -\beta_l\} \text{ and } |\mathcal{I}| = 1. \tag{1.28}$$

Example 2.14, discussed below, provides an example where (1.27) fails when $\Upsilon_0 = \{\beta_l, -\beta_l\}$, but $|\mathcal{I}| = 2$.

In Remark 2.13 we discuss how Theorems 2.11 and 2.12 can be applied to problems derived from the linearized compressible 2D Euler equations.

1.3 Multiple amplification and optimality of the estimates

The literature on geometric optics for weakly stable hyperbolic boundary problems contains several examples of *first-order* amplification: roughly, oscillatory data (F, G) of a given amplitude and wavelength ϵ yields a solution whose amplitude is larger by a factor of $\frac{1}{\epsilon}$ (for example, [MA88, MR83, AM87, CG10, CGW14, CW17]. The estimate (1.22) in cases where $\mathbb{E} \geq 1$ suggests that multiple amplification by a factor $\frac{1}{\epsilon^{\mathbb{E}+1}}$ may occur.[18]

In section 5 we study a weakly stable 3×3 problem (2.35) for which $\Upsilon_0 = \{\beta_l, -\beta_l\}$. There are three phases

$$\phi_i(t, x) = \beta_l \cdot (t, x_1) + \omega_i(\beta_l)x_2,\ i = 1, 2, 3, \tag{1.29}$$

with ϕ_2, ϕ_3 incoming and ϕ_1 outgoing, and the ϕ_j have the property that they exhibit exactly one *resonance* (defined in (4.39)):

$$-2\phi_2 + \phi_3 = -\phi_1. \tag{1.30}$$

The oscillatory coefficient in the problem is given by $d(\theta_{in}) = e^{i\theta_3}M$, where M is a particular constant matrix. In our third main result, Example 2.14, we construct and rigorously justify high order approximate solutions to the system (2.35) which exhibit instantaneous double amplification, that is, amplification by a factor of $\frac{1}{\epsilon^2}$ evident at any time $t > 0$. The amplified solution consists of a wave in the boundary that travels along a characteristic of the Lopatinski determinant and which "radiates" doubly amplified waves into the interior which travel along characteristics corresponding to the two incoming phases.[19]

In Remark 5.14 we explain that successively adding appropriately chosen terms to $d(\theta_{in})$ yields problems that are expected to exhibit instantaneous 3rd order, 4th order, 5th order, ... amplification. Each such problem would exhibit the maximum order of amplification permitted by the estimate (1.22)

[18] Recall that the factors $|X_k|$ on the right in (1.22) already produce one order of amplification.

[19] The boundary amplitude equation governing propagation of the wave in the boundary is a transport equation with a nonlocal zero-order term (5.75) involving a Fourier multiplier $m(D_{\theta_0})$.

when $\mathbb{E}$ is computed by the formula (2.30) of Theorem 2.11, and would thus further demonstrate the optimality of that estimate.

Proposition 4.6 considers problems with resonances in the "bad cases" for which the quantity $\Omega_{i,j}$ (defined below in (1.38)) is rational and lies in $(0, \infty)$ or $(-\infty, -1)$.[20] This proposition, taken together with our examples of multiple amplification, indicates that for such problems there is no hope of proving an estimate like (1.22) with finite $\mathbb{E}$ when the Fourier spectrum of $d(\theta_{in})$ is an arbitrary infinite subset of $\mathbb{Z}$. That remains true even when the spectrum of $d(\theta_{in})$ is restricted to be purely positive.

Our example of double amplification confirms that there are situations in which some of the global amplification factors $\mathbb{D}(\epsilon, k, k-r)(\zeta)$ are indeed large, that is, equal to $\frac{C_5|r|^3}{\epsilon\gamma}$, on $\zeta-$sets of positive measure. We do not see how to be sure of this without such examples. Section 6 contains more discussion related to this point.

Finally, observe that any estimate like (1.22) for which $\mathbb{E} = 0$, such as the estimate (2.32) in Theorem 2.12, is optimal. Simple examples show that the amplifying factors $|X_k|$ on the right of (1.22) are unavoidable in weakly stable problems like (1.1) even when $\mathcal{D} = 0$ [CG10].

Remark 1.2. *1) The results of section 3 show that the factors $\mathbb{D}(\epsilon, k, k-r)$ are determined just by our assumptions on $(L(\partial), B)$; they are* independent *of the choice of the oscillatory factor $\mathcal{D}(\theta_{in})$.*

2) If $\mathcal{D}$ is replaced by $\epsilon\mathcal{D}$ in (1.1)*, then V_{k-r-t} is replaced by ϵV_{k-r-t} on the right in the iteration estimate* (1.14)*, and this leads (by a simple argument* not *involving iteration of* (1.14)*) to a final estimate like* (1.22) *where $\mathbb{E} = 0$; multiple amplification does not occur.*[21] *The properties of the spectrum of $\mathcal{D}$, the assumptions involving the quantity $\Omega_{i,j}$ made in Theorems 2.11 and 2.12, the presence or absence of resonances, and the choice of ϕ_{in} or ϕ_{out} in the argument of $\epsilon\mathcal{D}$ are all irrelevant to the estimate in this case. Multiple instantaneous amplification is a phenomenon associated with large oscillatory coefficients.*

1.4 Remarks on the proofs.

We conclude with some informal remarks intended to give more insight into the proofs.

Iteration and cascade estimates. Consider first the formulas (3.38), (3.39) for solutions $w_k^+(x_2, \zeta)$, $w_k^-(x_2, \zeta)$ to the system obtained by diagonalizing the singular transformed problem (1.13) near one of the bad directions $\beta \in \Upsilon^0$. For a given (ϵ, k) these formulas are valid for ζ such that $X_k := \zeta + k\frac{\beta_l}{\epsilon}$ lies in a small enough conic neighborhood $\Gamma_\delta(\beta)$ of β.[22] The functions of ζ denoted by $\omega_j(\epsilon, k; \beta)$ that appear in (3.37) are related to the eigenvalues $\omega_j(X_k)$ of $\mathcal{A}(X_k)$ ($\mathcal{A}$ as in (1.13)) by the slightly abusive equation

$$\omega_j(\epsilon, k; \beta)(\zeta) = \omega_j(X_k) \text{ for } X_k \in \Gamma_\delta(\beta). \tag{1.31}$$

The "most dangerous" terms in the expressions for $w_k^\pm$ are the terms in the sum appearing in the second line of the expression for $w_k^-(x_2, \zeta)$, (3.39). The matrix $[Br_-(\epsilon, k)(\zeta)]^{-1}$ that appears there is large for

[20]For the resonance (1.30) we have $\Omega_{1,2} = 1$.

[21]Such an estimate in this case already follows by the simultaneous diagonalization argument of [CGW14].

[22]Here $\beta \in \Upsilon^0$ may or may not equal β_l, where $\beta_l \in \Upsilon^0$ is the particular bad direction that appears in (1.2); β_l is fixed throughout the paper. This same β_l appears in the definition $X_k := \zeta + k\frac{\beta_l}{\epsilon}$. When ζ is large compared to $k\frac{\beta_l}{\epsilon}$, X_k can of course lie far from the β_l direction.

X_k near β and satisfies[23]

$$|[Br_-(\epsilon,k)(\zeta)]^{-1}| \lesssim |\Delta(X_k)|^{-1} \lesssim \frac{|X_k|}{\gamma} \text{ for } \gamma>0. \tag{1.32}$$

With the normalizations chosen in the next section (which imply (3.40), e.g.), one sees from (3.39) that to control the $r-$th term in the sum one must control quantities like

$$\frac{1}{\Delta(X_k)}\int_0^\infty e^{i\omega_i(X_k)(-s)+ir\frac{\omega_N(\beta_l)}{\epsilon}s}\alpha_r w^-_{k-r,j}(s,\zeta)ds,\ i\in\mathcal{O},\ \ j,N\in\mathcal{I} \tag{1.33}$$

where $w^-_{k-r,j}$ is the $j-$th component of w^-_{k-r}.

Consider the case when X_k lies in a small conic neighborhood of β_l, $\Gamma_\delta(\beta_l)$. We can try to control the factor $|\Delta(X_k)|^{-1}$ in (1.33) by doing an integration by parts in the oscillatory integral. An $s-$derivative will fall on $w^-_{k-r,j}$, and we suppose for these remarks that $X_{k-r}\in\Gamma_\delta(\beta_l)$ in order to use the equation for $w^-_{k-r,j}$ given by $(3.36)_{k-r}$. Since $\partial_s w^-_{k-r,j}$ can grow like $|X_k|$, there is no sure gain if we do the integration by parts directly on (1.33). Instead, we set

$$w^{*,-}_{k-r,j}(x_2,\zeta) := e^{-i\omega_j(X_{k-r})x_2}w^-_{k-r,j},\text{ noting that } \partial_{x_2}w^{*,-}_{k-r,j}=e^{-i\omega_j(X_{k-r})x_2}h^-_{k-r,j}, \tag{1.34}$$

where $h^-_{k-r,j}$ is the j-component of the right side of $(3.36)_{k-r}$. We then rewrite (1.33) as

$$\begin{aligned}&\frac{1}{\Delta(X_k)}\int_0^\infty e^{i\omega_i(X_k)(-s)+ir\frac{\omega_N(\beta_l)}{\epsilon}s}e^{i\omega_j(X_{k-r})s}\alpha_r w^{*,-}_{k-r,j}(s,\zeta)ds=\\ &\frac{1}{\Delta(X_k)}\int_0^\infty e^{-iE_{i,j}(\epsilon,k,k-r)s}\alpha_r w^{*,-}_{k-r,j}(s,\zeta)ds,\text{ where } E_{i,j}(\epsilon,k,k-r)=\omega_i(X_k)-r\frac{\omega_N(\beta_l)}{\epsilon}-\omega_j(X_{k-r}).\end{aligned} \tag{1.35}$$

We show in Proposition 3.10 and its refinements, Propositions 3.21 and 3.22, that it "frequently" happens that

$$|E_{i,j}(\epsilon,k,k-r)(\zeta)|\geq C_3\frac{|X_k|}{|r|}\text{ or } |E_{i,j}(\epsilon,k,k-r)|\geq C_3|X_{k-r}|,\text{ for a } C_3 \text{ independent of } (\epsilon,\zeta,k,r). \tag{1.36}$$

It is clear from (1.32) that when the first possibility in (1.36) holds, an integration by parts can be used to control the factor $|\Delta(X_k)|^{-1}$ in (1.35). It is less clear but true that when the second possibility holds, one can also control $|\Delta(X_k)|^{-1}$; see step 7 of the proof of Proposition 3.18. If for some $(i,j)\in\mathcal{O}\times\mathcal{I}$ and (ϵ,ζ,k,r) the alternative (1.36) fails to hold, then we must define the global amplification factor $\mathbb{D}(\epsilon,k,k-r)(\zeta)$ to be $\frac{C_5|r|^3}{\epsilon\gamma}$ for that particular choice of (ϵ,ζ,k,r).[24]

One must also consider cases where $X_k\in\Gamma_\delta(\beta)$ for $\beta\in\Upsilon^0\setminus\{\pm\beta_l\}$. In these cases the quantity $\frac{\Delta(X_{k-r})}{\Delta(X_k)}$ is useful for controlling the large factor $|\Delta(X_k)|^{-1}$. In Proposition 3.11 we show that it "usually" happens that

$$\left|\frac{\Delta(X_{k-r})}{\Delta(X_k)}\right|\leq C_1r^2,\text{ for a } C_1 \text{ independent of } (\epsilon,\zeta,k,r). \tag{1.37}$$

[23]The function $\Delta(\zeta):=\det Br_-(\zeta)$ (Definition 2.8) is the Lopatinski determinant. The uniform Lopatinski condition fails at β precisely when $\Delta(\beta)=0$.

[24]In the iteration estimate (1.14), for a given $r\in\mathbb{Z}\setminus 0$ most of the terms in the inner sum $\sum_{t\in\mathbb{Z}}$ come from terms like $e^{-i\omega_j(X_{k-r})x_2}h^-_{k-r,j}$ as in (1.34), which result from the integration by parts.

If for some (ϵ, ζ, k, r) the estimate (1.37) fails to hold, then we must define the global amplification factor $\mathbb{D}(\epsilon, k, k-r)(\zeta) = \frac{C_5|r|^3}{\epsilon\gamma}$ for that particular choice of (ϵ, ζ, k, r). We restrict attention to two space dimensions partly in order to control quotients like the one in (1.37) (Remark 3.7).[25]

In order to carry out the cascade estimates, we must understand how "often" (1.36) and (1.37) can fail. This work is done in the Propositions listed in the above two paragraphs, as well as in Proposition 4.3. For example, Proposition 3.21 shows that when

$$\Omega_{i,j} := \frac{\omega_i(\beta_l) - \omega_N(\beta_l)}{\omega_j(\beta_l) - \omega_i(\beta_l)} \in (-1, 0), \tag{1.38}$$

then (1.36) holds for "most" (k, r). Proposition 4.3 counts for any given (ϵ, ζ) how many of the amplification factors in products like (1.21) can be large. It turns out that under the assumptions of Theorems 2.11 and 2.12, no more than $\mathbb{E}$ factors can be large, where $\mathbb{E}$ is specified in those theorems.

Construction of approximate solutions. In section 5 we construct approximate solutions exhibiting double amplification to the problem (2.35). The approximate solutions have the form

$$u_a^\epsilon(t,x) = \sum_{k=-1}^{J} \epsilon^k U_k(t, x, \frac{\Phi}{\epsilon}), \quad U_k(t,x,\theta) = \sum_{\alpha\in\mathbb{Z}^3} U_{k,\alpha}(t,x)e^{i\alpha\theta}, \tag{1.39}$$

where $\Phi(t,x) = (\phi_1, \phi_2, \phi_3)$ is a triple of resonant phases, and the profiles $U_k(t,x,\theta)$ are $2\pi-$periodic with respect to $\theta = (\theta_1, \theta_2, \theta_3)$. An extra difficulty in the construction of approximate solutions exhibiting multiple (as opposed to single) amplification is the higher degree of coupling among the profile equations. For example, to determine the trace of the leading order profile U_{-1}, we must now solve two coupled boundary amplitude equations, (5.32)(a),(b), instead of just one boundary amplitude equation as in the case of single amplification. The equations (5.32)(a),(b) involve in turn higher unknown amplitudes (respectively, U_0, U_1).

To cope with this high degree of coupling, we introduce the following device. We make a list in section 5.3 of the possible modes that we *expect* to appear in the various profiles; that is, for each U_k in (1.39), we specify the α for which $U_{k,\alpha}$ might possibly be nonzero.[26] The list is at first just a reasonable guess that takes into account the boundary data in (2.35), the single resonance (2.36), the profile equations, and what we already know about the exact solution. We then make two assumptions on which we base the construction of the profiles: (a) profiles U_k exist which satisfy all the profile equations; (b) the only possible nonzero modes of these profiles are those which appear in our list. It is not clear at first that these assumptions are *consistent*. However, by making these assumptions we are able to construct profiles that satisfy the profile equations and whose nonzero modes manifestly lie in our list. Thus, the construction itself verifies the consistency and correctness of the two assumptions. A key advantage of this approach is that it allows us to decouple the equations by solving for *individual* modes of profiles in the appropriate order, starting with the low modes (that is, modes for which $|\alpha|$ is small.) We are not concerned about uniqueness of the profiles, because we know that the exact solution is unique, and we show in Example 2.14 that the approximate solution is $O(\epsilon^\infty)$ close in L^∞ to the exact solution.

Notations 1.3. *1) If $f(\tau, \eta, \epsilon, k, \gamma)$ is a function of $(\tau, \eta, \epsilon, k, \gamma) \in \mathcal{D}$ for some domain $\mathcal{D}$, the statement $f \sim 1$ means that there exist positive constants A_1, A_2 independent of $(\tau, \eta, \epsilon, k, \gamma) \in \mathcal{D}$ such that*

$$A_1 \leq |f(\tau, \eta, \epsilon, k, \gamma)| \leq A_2 \text{ on } \mathcal{D}.$$

[25]See also Remark 2.10 for more about the restriction to two space dimensions.

[26]In section 5.3 we don't actually list the various α, but the list we give is equivalent to a specification of the α.

2) If g is another such function, the statement $f \lesssim g$ means that there exists a positive constant C independent of $(\tau, \eta, \epsilon, k, \gamma) \in \mathcal{D}$ such that

$$|f| \leq C|g| \text{ on } \mathcal{D}.$$

3) The constants C, C_1, C_2, K, M, etc. that appear frequently in the estimates below are always independent of the important parameters $(\tau, \eta, \epsilon, k, \gamma)$.

4) If S is a finite set, we denote the cardinality of S by $|S|$.

5) For $n \in \mathbb{N}$ we denote the $n \times n$ identity matrix by I_n.

6) We sometimes denote the norm on $L^2(\Omega \times \mathbb{T})$ by $|U|_{L^2(t,x,\theta)}$ (and use similar notation for other spaces), when the domain of the variables (t, x, θ) is clear from the context.

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2 Assumptions and main results

We consider the problem

$$\begin{aligned}
&\partial_t u + B_1 \partial_{x_1} u + B_2 \partial_{x_2} u + \mathcal{D}\left(\frac{\phi_{in}}{\epsilon}\right) u = F(t, x, \frac{\phi_0}{\epsilon}) \text{ in } x_2 > 0 \\
&Bu = G(t, x_1, \frac{\phi_0}{\epsilon}) \text{ on } x_2 = 0 \qquad (2.1)\\
&u = 0 \text{ in } t < 0.
\end{aligned}$$

Here the B_j are constant $N \times N$ matrices, B_2 is invertible, and the boundary phase is taken to be $\phi_0(t, x_1) = \beta_l \cdot (t, x_1)$, where $\beta_l = (\sigma_l, \eta_l) \in \mathbb{R}^2 \setminus 0$ is one of the directions where the uniform Lopatinski condition (Definition 2.5) fails. The matrix $\mathcal{D}(\theta_{in})$ and functions $F(t, x, \theta)$, $G(t, x_1, \theta)$ are respectively $2\pi-$periodic in θ_{in} and θ. Also,

$$\phi_{in}(t, x) = \phi_0(t, x_1) + \omega_N(\beta_l) x_2 \qquad (2.2)$$

is one of the incoming phases (see Definition 2.6). For convenience we take F and G to be zero in $t < 0$.

Assumption 2.1 (Strict hyperbolicity)**.** *The B_j are real matrices, and there exist real valued functions $\lambda_j(\eta, \xi)$, $j = 1, \ldots, N$ that are analytic on $\mathbb{R}^2 \setminus 0$ and homogeneous of degree one such that*

$$\det(\sigma I + B_1 \eta + B_2 \xi) = \prod_{k=1}^{N} (\sigma + \lambda_k(\eta, \xi)) \text{ for all } (\eta, \xi) \in \mathbb{R}^2 \setminus 0. \qquad (2.3)$$

Moreover, we have

$$\lambda_1(\eta, \xi) < \lambda_2(\eta, \xi) < \cdots < \lambda_N(\eta, \xi) \text{ for all } (\eta, \xi) \in \mathbb{R}^2 \setminus 0. \qquad (2.4)$$

We rewrite (2.1) as

$$
\begin{aligned}
&D_{x_2} u + A_0 D_t u + A_1 D_{x_1} u - i B_2^{-1} \mathcal{D}\left(\frac{\phi_{in}}{\epsilon}\right) u = F(t, x, \frac{\phi_0}{\epsilon}) \\
&Bu = G(t, x_1, \frac{\phi_0}{\epsilon}) \text{ on } x_2 = 0 \qquad (2.5) \\
&u = 0 \text{ in } t < 0,
\end{aligned}
$$

where $A_0 = B_2^{-1}$, $A_1 = B_2^{-1} B_1$, and F has been modified in an unimportant way. Let us introduce the matrix

$$
\mathcal{A}(\tau, \eta) = -(A_0 \tau + A_1 \eta), \quad (\tau, \eta) = (\sigma - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}, \tag{2.6}
$$

and define the following sets of frequencies:

$$
\begin{aligned}
\Xi &:= \Big\{ \zeta := (\sigma - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R} \setminus (0,0) : \gamma \geq 0 \Big\}, & \Sigma &:= \Big\{ \zeta \in \Xi : \sigma^2 + \gamma^2 + \eta^2 = 1 \Big\}, \\
\Xi_0 &:= \Big\{ (\sigma, \eta) \in \mathbb{R} \times \mathbb{R} \setminus (0,0) \Big\} = \Xi \cap \{\gamma = 0\}, & \Sigma_0 &:= \Sigma \cap \Xi_0 .
\end{aligned}
$$

The hyperbolic region and the glancing set are defined as follows.

Definition 2.2. *a) The hyperbolic region $\mathcal{H}$ is the set of all $(\sigma, \eta) \in \Xi_0$ such that the matrix $\mathcal{A}(\sigma, \eta)$ is diagonalizable with real eigenvalues.*

b) Let $\mathbf{G}$ denote the set of all $(\sigma, \eta, \xi) \in \mathbb{R} \times \mathbb{R}^2$ such that $(\eta, \xi) \neq 0$ and there exists an integer $k \in \{1, \ldots, N\}$ satisfying:

$$
\sigma + \lambda_k(\eta, \xi) = \frac{\partial \lambda_k}{\partial \xi}(\eta, \xi) = 0 .
$$

If $\pi(\mathbf{G})$ denotes the projection of $\mathbf{G}$ on the first 2 coordinates (in other words $\pi(\sigma, \eta, \xi) = (\sigma, \eta)$ for all (σ, η, ξ)), the glancing set $\mathcal{G}$ is $\mathcal{G} := \pi(\mathbf{G}) \subset \Xi_0$.

Assumption 2.3. *The matrix B_2 is invertible and has p positive eigenvalues, where $1 \leq p \leq N - 1$. The boundary matrix B is $p \times N$, real, and of rank p.*

Proposition 2.4. *[Kre70] Let Assumptions 2.1 and 2.3 be satisfied. Then for all $\zeta \in \Xi \setminus \Xi_0$, the matrix $i\mathcal{A}(\zeta)$ has no purely imaginary eigenvalue and its stable subspace $\mathbb{E}^s(\zeta)$ has dimension p. Furthermore, $\mathbb{E}^s$ defines an analytic vector bundle over $\Xi \setminus \Xi_0$ that can be extended as a continuous vector bundle over Ξ.*

For all $(\sigma, \eta) \in \Xi_0$, we let $\mathbb{E}^s(\sigma, \eta)$ denote the continuous extension of $\mathbb{E}^s(\zeta)$ to the point (σ, η). The analysis in [Mét00] shows that away from the glancing set $\mathcal{G} \subset \Xi_0$, $\mathbb{E}^s(\zeta)$ depends analytically on ζ, and the hyperbolic region $\mathcal{H}$ does not contain any glancing point.

Definition 2.5. *[Kre70] As before let p be the number of positive eigenvalues of B_2, and let*

$$
L(\partial) = \partial_t + B_1 \partial_{x_1} + B_2 \partial_{x_2} .
$$

The problem $(L(\partial), B)$ is said to be uniformly stable *or to satisfy the* uniform Lopatinski condition *(ULC) if*

$$
B : \mathbb{E}^s(\zeta) \longrightarrow \mathbb{C}^p \tag{2.7}
$$

is an isomorphism for all $\zeta \in \Sigma$. Similarly, we say $(L(\partial), B)$ satisfies the ULC on Ξ, (respectively, on a closed conic subset $\Gamma \subset \Xi$), if the map (2.7) *is an isomorphism on Σ (respectively, on the subset of Σ corresponding to Γ).*

We now fix a choice of $\beta = (\underline{\sigma}, \underline{\eta}) \in \mathcal{H}$. As a consequence of strict hyperbolicity there is a closed conic neighborhood $\Gamma_\delta^+(\beta)$ of β in Ξ with opening angle $\delta > 0$,

$$\Gamma_\delta^+(\beta) = \left\{ \zeta \in \Xi : \left| \frac{\zeta}{|\zeta|} - \frac{\beta}{|\beta|} \right| \leq \delta \right\},$$

such that $\mathcal{A}(\zeta)$ has N distinct eigenvalues $\omega_j(\zeta)$ and corresponding eigenvectors $R_j(\zeta)$ satisfying

$$\mathcal{A}(\zeta) R_j(\zeta) = \omega_j(\zeta) R_j(\zeta),\ j = 1, \ldots, N \text{ on } \Gamma_\delta^+(\beta). \tag{2.8}$$

The functions ω_j, R_j map $\Gamma_\delta^+(\beta)$ into $\mathbb{C}$, $\mathbb{C}^N$ respectively, are homogeneous of degree one, and are analytic in τ, C^∞ (in fact, real-analytic) in η. We also define normalized vectors

$$r_j(\zeta) := R_j(\zeta)/|R_j(\zeta)|, \tag{2.9}$$

which are merely C^∞ in (σ, γ, η).

To each root $\omega_j(\beta) = \underline{\omega}_j$ there corresponds a unique integer $k_j \in \{1, \ldots, N\}$ such that $\underline{\sigma} + \lambda_{k_j}(\underline{\eta}, \underline{\omega}_j) = 0$. We can then define the following real phases ϕ_j and their associated group velocities:

$$\forall\, j = 1, \ldots, N\,, \quad \phi_j(x) := \phi_0(t, y) + \underline{\omega}_j\, x_2\,, \quad \mathbf{v}_j := \nabla \lambda_{k_j}(\underline{\eta}, \underline{\omega}_j)\,. \tag{2.10}$$

Let us observe that each group velocity $\mathbf{v}_j$ is either incoming or outgoing with respect to the space domain $\mathbb{R}^2_+$: the last coordinate of $\mathbf{v}_j$ is nonzero. This property holds because β does not belong to the glancing set $\mathcal{G}$. For any $\beta \in \mathcal{H}$ there are exactly $N - p$ outgoing phases and (after relabelling if necessary) we denote the corresponding set of indices by $\mathcal{O} = \{1, \ldots, N - p\}$. The set of incoming indices is $\mathcal{I} = \{N - p + 1, \ldots, N\}$.

We can therefore adopt the following classification:

Definition 2.6. *The phase ϕ_j is incoming if the group velocity $\mathbf{v}_j$ is incoming (that is, when $\partial_\xi \lambda_{k_j}(\underline{\eta}, \underline{\omega}_j) > 0$), and it is outgoing if the group velocity $\mathbf{v}_j$ is outgoing ($\partial_\xi \lambda_{k_j}(\underline{\eta}, \underline{\omega}_j) < 0$). If the phase ϕ_j is incoming (resp., outgoing), we shall also refer to the corresponding frequency $\underline{\omega}_j$ as incoming (resp., outgoing).*

The ω_j are real-valued for $\gamma = 0$ and can be divided into two groups according as $j \in \mathcal{O}$ or $\mathcal{I}$. There exists a constant $c > 0$ such that for $\zeta \in \Gamma_\delta^+(\beta)$

$$\begin{aligned} &\operatorname{Im} \omega_j(\zeta) \leq -c\gamma \text{ for } j \in \mathcal{O} \\ &\operatorname{Im} \omega_j(\zeta) \geq c\gamma \text{ for } j \in \mathcal{I}. \end{aligned} \tag{2.11}$$

We define the $N \times (N - p)$ matrix $r_+(\zeta)$ and the $N \times p$ matrix $r_-(\zeta)$ by

$$r_+ = \begin{pmatrix} r_1 & r_2 & \ldots & r_{N-p} \end{pmatrix}, \qquad r_- = \begin{pmatrix} r_{N-p+1} & \ldots & r_N \end{pmatrix} \text{ on } \Gamma_\delta^+(\beta). \tag{2.12}$$

Similarly, define $R_\pm(\zeta)$ using the unnormalized eigenvectors $R_j(\zeta)$.

Next we introduce the $N \times N$ matrix

$$S(\zeta) = \begin{pmatrix} r_+(\zeta) & r_-(\zeta) \end{pmatrix} \text{ on } \Gamma_\delta^+(\beta). \tag{2.13}$$

Having fixed the column vectors $r_j(\zeta)$, we define an $(N-p)\times N$ matrix $\ell_+(\zeta)$ and a $p\times N$ matrix $\ell_-(\zeta)$ such that

$$S^{-1}(\zeta) = \begin{pmatrix} \ell_+(\zeta) \\ \ell_-(\zeta) \end{pmatrix} \text{ on } \Gamma^+_\delta(\beta). \tag{2.14}$$

The rows of $S^{-1}(\zeta)$ are given by row vectors $\ell_j(\zeta)$, $j = 1,\dots,N$, and these satisfy $\ell_j(\zeta)\sim 1$.

The following well-known proposition, proved in [CG10], gives a useful decomposition of $\mathbb{E}^s(\zeta)$.

Proposition 2.7. *For $\zeta\in\Gamma^+_\delta(\beta)$ the stable subspace $\mathbb{E}^s(\zeta)$ admits the decomposition*

$$\mathbb{E}^s(\zeta) = \bigoplus_{j\in\mathcal{I}} \operatorname{span} r_j(\zeta), \tag{2.15}$$

and the vectors $r_j(\beta)$ can be (and are) taken to be real vectors.

Next we define the Lopatinski determinant and recall some facts from [Wil20].

Definition 2.8. *For $\zeta\in\Gamma^+_\delta(\beta)$ define the analytic Lopatinski determinant*

$$\Delta_a(\zeta) = \det BR_-(\zeta). \tag{2.16}$$

and the normalized Lopatinski determinant

$$\Delta(\zeta) = \det Br_-(\zeta). \tag{2.17}$$

Observe that $\Delta(\zeta)$ is C^∞ in ζ and positively homogeneous of degree 0 on $\Gamma^+_\delta(\beta)$. Moreover, the map (2.7) fails to be an isomorphism at $\zeta\in\Gamma^+_\delta(\beta)$ if and only if $\Delta(\zeta)=0$.

We can now formulate our weak stability assumption on the problem $(L(\partial), B)$.

Assumption 2.9.

- *For all $\zeta\in\Xi\setminus\Xi_0$, $\mathrm{Ker}\ B\cap\mathbb{E}^s(\zeta)=\{0\}$.*
- *The set $\Upsilon_0 := \{\zeta\in\Sigma_0 : \mathrm{Ker}\ B\cap\mathbb{E}^s(\zeta)\neq\{0\}\}$ is nonempty and included in the hyperbolic region $\mathcal{H}$.*
- *For all $\beta\in\Upsilon_0$ there exists a neighborhood $\Gamma^+_\delta(\beta)$ as above on which functions ω_j, R_j, Δ_a are defined and we have*

$$\Delta_a(\beta)=0 \text{ and } \partial_\tau\Delta_a(\beta)\neq 0. \tag{2.18}$$

Remark 2.10. *1). Using* (2.18) *and the implicit function theorem, and after reducing $\delta>0$ if necessary, we may write*

$$\Delta_a(\tau,\eta) = (\tau - g(\eta))H_+(\tau,\eta) \text{ on } \Gamma^+_\delta(\beta), \tag{2.19}$$

where g and H_+ inherit the obvious regularity from Δ_a, g is homogeneous of degree one, and H_+ is homogeneous of degree $p-1$ and nonvanishing on $\Gamma^+_\delta(\beta)$. Since $d=2$ the function g is in fact linear

$$g(\eta) = c_+\eta \text{ for some } c_+ = c_+(\beta)\in\mathbb{R}. \tag{2.20}$$

2). Since $\Delta(\zeta)=\frac{\Delta_a(\zeta)}{\prod_{j=N-p+1}^{N}|R_j(\zeta)|}$, *we obtain from* (2.19), (2.20)*:*

$$\Delta(\tau,\eta)=(\tau-c_+\eta))h_+(\tau,\eta)\ \text{on}\ \Gamma^+_\delta(\beta), \tag{2.21}$$

where $h_+(\tau,\eta)$ *is homogenous of degree* -1 *and nonvanishing on* $\Gamma^+_\delta(\beta)$.

3.)Using Assumption 2.9, the compactness of Υ_0, *and the analyticity of* Δ_a *one can deduce as in [BGS07], section 8.3, that* Υ_0 *is a finite set in the case* $d=2$ *that we are considering:*

$$\Upsilon_0=\{\beta_j,\ j=1,\dots,M_0\}. \tag{2.22}$$

Proposition 3.1 below shows that M_0 *is even.*

We can now state our main result for problems with oscillatory coefficients having only positive spectrum, the case of one-sided cascades. There is, of course, an exactly parallel result for coefficients with only negative spectrum.

Consider the singular problem (1.7) for $U(t,x,\theta)$ or the equivalent transformed problem for $V_k(x_2,\zeta):=\widehat{U_k}(\zeta,x_2)$:

$$\begin{aligned}&(a)D_{x_2}V_k-\mathcal{A}(X_k)V_k=i\sum_{r\in\mathbb{Z}\setminus 0}e^{ir\frac{\omega_N(\beta_l)}{\epsilon}x_2}B_2^{-1}\widehat{\mathcal{D}}(r)V_{k-r}+\widehat{F_k}(x_2,\zeta)\\&(b)BV_k=\widehat{G_k}(\zeta)\text{ on }x_2=0.\end{aligned} \tag{2.23}$$

Theorem 2.11 (Energy estimate: $\mathcal{D}$ has positive spectrum only)**.**

(a). Consider solutions $U(t,x,\theta)$ *of the singular system* (1.7) *with forcing terms* $F=\sum_{k\in\mathbb{Z}}F_k(t,x)e^{ik\theta}$, $G=\sum_{k\in\mathbb{Z}}G_k(t,x_1)e^{ik\theta}$ *in* $H^1(\Omega\times\mathbb{T})$, $H^1(\mathbb{R}^2\times\mathbb{T})$ *respectively, under Assumptions 2.1, 2.3, 2.9. Assume the* $N\times N$ *matrices* $\widehat{\mathcal{D}}(r)$ *in* (2.23) *satisfy*

$$\widehat{\mathcal{D}}(r)=0\ \text{for}\ r\le 0,\ \ |\widehat{\mathcal{D}}(r)|\lesssim|r|^{-(M+3)}\ \text{for some}\ M\ge 2. \tag{2.24}$$

For $i\in\mathcal{O}$, $j\in\mathcal{I}\setminus\{N\}$ *let* [27]

$$\Omega_{i,j}:=\frac{\omega_i(\beta_l)-\omega_N(\beta_l)}{\omega_j(\beta_l)-\omega_i(\beta_l)}, \tag{2.25}$$

and suppose

$$\Omega_{i,j}\in(-1,0)\ \text{for}\ i\in\mathcal{O},j\in(\mathcal{I}\setminus\{N\}). \tag{2.26}$$

Define the natural number $\mathbb{E}$ *by*

$$\mathbb{E}=|\mathcal{O}|(|\mathcal{I}|-1)+2\left(\frac{|\Upsilon_0|}{2}-1\right). \tag{2.27}$$

Then there exist positive constants γ_0, K *such that for* $0<\epsilon<\epsilon_0$ *and* $\gamma\ge\gamma_0$ *we have*

$$|U^\gamma|_{L^2(t,x,\theta)}+\left|\frac{U^\gamma(0)}{\sqrt{\gamma}}\right|_{L^2(t,x_1,\theta)}\le\frac{K}{(\epsilon\gamma)^{\mathbb{E}}}\left[\frac{1}{\gamma^2}\left(\sum_{k\in\mathbb{Z}}\left||X_k|\widehat{F_k}\right|^2_{L^2(x_2,\sigma,\eta)}\right)^{1/2}+\frac{1}{\gamma^{3/2}}\left(\sum_{k\in\mathbb{Z}}\left|\widehat{G_k}|X_k|\right|^2_{L^2(\sigma,\eta)}\right)^{1/2}\right]. \tag{2.28}$$

[27] Here the $\omega_j(\beta_l)$ are the distinct real eigenvalues of $\mathcal{A}(\beta_l)$; recall (2.8).

Equivalently,

$$|U^\gamma|_{L^2(t,x,\theta)} + \left|\frac{U^\gamma(0)}{\sqrt{\gamma}}\right|_{L^2(t,x_1,\theta)} \leq \frac{K}{(\epsilon\gamma)^{\mathbb{E}}}\left[\frac{1}{\gamma^2}|\Lambda_D F^\gamma|_{L^2(t,x,\theta)} + \frac{1}{\gamma^{3/2}}|\Lambda_D G^\gamma|_{L^2(t,x_1,\theta)}\right], \tag{2.29}$$

where $U^\gamma := e^{-\gamma t}U$, $U^\gamma(0)$ is the trace on $x_2 = 0$, and Λ_D is the singular operator associated to $|X_k|$.[28]

(b) Under Assumptions 2.1, 2.3, 2.9 suppose now that the oscillatory coefficient in (1.7) *has* finite *positive spectrum, that is, suppose there exists $P \in \mathbb{N}$ such that $\widehat{\mathcal{D}}(r) = 0$ for all but P distinct choices of $r \in \mathbb{N}$. We now make* no assumption *on the numbers $\Omega_{i,j}$. Then the estimate* (2.28) *holds with*

$$\mathbb{E} = P|\mathcal{O}|(|\mathcal{I}| - 1) + 2\left(\frac{|\Upsilon_0|}{2} - 1\right). \tag{2.30}$$

Next we state our main result for problems with oscillatory coefficients having both positive and negative spectrum, the case of two-sided cascades.

Theorem 2.12 (Energy estimate: $\mathcal{D}$ has positive and negative spectrum)**.** *Consider solutions $U(t,x,\theta)$ of the singular system* (1.7) *with forcing terms $F = \sum_{k\in\mathbb{Z}} F_k(t,x)e^{ik\theta}$, $G = \sum_{k\in\mathbb{Z}} G_k(t,x_1)e^{ik\theta}$ in $H^1(\Omega\times\mathbb{T})$, $H^1(\mathbb{R}^2\times\mathbb{T})$ respectively. Suppose the coefficients $\widehat{\mathcal{D}}(r)$ in* (2.23) *satisfy $\widehat{\mathcal{D}}(0) = 0$ and $|\widehat{\mathcal{D}}(r)| \lesssim |r|^{-(M+1)}$ for some $M \geq 2$.*

As before we make the structural Assumptions 2.1, 2.3, 2.9, but now we add *the assumption that*

$$\Upsilon_0 = \{\beta_l, -\beta_l\} \text{ and } \mathcal{I} = \{N\} \ (\text{thus}, \mathcal{O} = \{1, \ldots, N-1\}). \tag{2.31}$$

Then there exist positive constants ϵ_0, γ_0, K such that for $0 < \epsilon \leq \epsilon_0$ and $\gamma \geq \gamma_0$ we have

$$|U^\gamma|_{L^2(t,x,\theta)} + \left|\frac{U^\gamma(0)}{\sqrt{\gamma}}\right|_{L^2(t,x_1,\theta)} \leq K\left[\frac{1}{\gamma^2}\left(\sum_{k\in\mathbb{Z}}\left||X_k|\widehat{F_k}\right|^2_{L^2}\right)^{1/2} + \frac{1}{\gamma^{3/2}}\left(\sum_{k\in\mathbb{Z}}\left|\widehat{G_k}|X_k|\right|^2_{L^2(\sigma,\eta)}\right)^{1/2}\right]. \tag{2.32}$$

Remark 2.13. *1) Suppose that $\Omega_{i,j}$ defined by* (2.25) *lies in* $(-\infty,-1)$*. Then by interchanging ω_N and ω_j (recall $j, N \in \mathcal{I}$) we obtain a quotient that lies in* $(-1,0)$*.*

2). In Appendix B of [CGW14] we consider the linearized compressible Euler equations in two space dimensions (a 3×3 system) obtained by linearizing at a given specific volume $v > 0$ and a subsonic incoming velocity $(0,u)$, $0 < u < c$. We choose a frequency β_l in the hyperbolic region which yields distinct eigenvalues $\omega_i(\beta_l)$, $i = 1,2,3$ such that ω_2, ω_3 are incoming and ω_1 is outgoing, so $|\mathcal{I}| = 2$. Taking the boundary matrix $B = \begin{pmatrix} 0 & v & 0 \\ u & 0 & v\end{pmatrix}$, we check that the weak stability assumptions 2.1, 2.3, 2.9 are satisfied with $\Upsilon^0 = \{\beta_l, -\beta_l\}$. Using the formulas for ω_j given there, one can verify

$$\frac{\omega_1 - \omega_2}{\omega_3 - \omega_1} \in (-1,0). \tag{2.33}$$

Thus, if we add an oscillatory zero-order term $\mathcal{D}(\phi_2/\epsilon)$ (with positive spectrum) to the linearized operator, we obtain a problem like (2.1) *to which part (a) of Theorem 2.11 applies.*

[28] We define $\Lambda_D G^\gamma := \sum_k \int e^{i(\sigma t + \eta x_1 + k\theta)}|X_k|\,\widehat{G_k}(\tau - i\gamma, \eta)d\sigma d\eta$.

3) Similarly, section 5.3 of [CG10] considers the linearized compressible Euler equations obtained by linearizing at $v > 0$ and a subsonic outgoing velocity $(0, u)$, $-c < u < 0$. The choices made there of β_l and a 1×3 boundary matrix B allow Theorem 2.12 to be applied; in this case $|\mathcal{I}| = 1$, $\Upsilon^0 = \{\beta_l, -\beta_l\}$.

4) In Theorem 2.12 or in any case of Theorem 2.11 where $\mathbb{E} = 0$, we can allow the oscillatory coefficient $\mathcal{D}$ to have nonzero mean, $\widehat{\mathcal{D}}(0) \neq 0$. In that case equation (2.23) is modified to be

$$D_{x_2} V_k - \mathcal{A}(X_k) V_k - i B_2^{-1} \widehat{\mathcal{D}}(0) V_k = i \sum_{r \in \mathbb{Z} \setminus 0} \alpha_r e^{ir \frac{\omega_N(\beta_l)}{\epsilon} x_2} B_2^{-1} \widehat{\mathcal{D}}(r) V_{k-r} + \widehat{F_k}(x_2, \zeta). \tag{2.34}$$

One can then use an argument in the proof of [CGW14], Proposition 2.4 to simultaneously diagonalize the second and third terms on the left of (2.34). This produces an error term $r_{-1}(X_k)V_k$, where r_{-1} is homogeneous of degree -1, which can be treated as part of the interior forcing and absorbed in the final estimate by taking γ large. The simultaneous diagonalization process replaces $\xi_\pm(X_k)$ in (3.36), (3.37) by $\xi_\pm(X_k) + r_{0_\pm}(X_k)$, where $r_{0_\pm}$ is homogeneous of degree zero. This change necessitates some straightforward, minor changes in the proof of the iteration estimate. When $\mathbb{E} \neq 0$ this argument does not work; the error term $r_{-1}(X_k)V_k$ cannot be treated as forcing and absorbed.

*5) For the problem (1.1) with an oscillatory coefficient $\mathcal{D}(\frac{\phi_{out}}{\epsilon})$ instead of $\mathcal{D}(\frac{\phi_{in}}{\epsilon})$, one can prove an analogue of Theorem 2.11, but with $\mathbb{E} = (|\mathcal{O}|-1)|\mathcal{I}| + 2(\frac{\Upsilon_0}{2} - 1) + 1$ now. This altered formula is optimal for the 2×2 example in section 4.4 of [CG10], and reflects the fact that some arguments in steps **7** and **8** of Proposition 3.18 fail in this case. Since $\mathbb{E}$ cannot be 0 now, we do not have an analogue of Theorem 2.12 in this case. Another reason for our focus on the $\mathcal{D}(\frac{\phi_{in}}{\epsilon})$ case is its greater relevance to the Mach stem problem; this is explained in [CW17].*

In section 5 we study a 3×3, strictly hyperbolic WR problem of the form

$$\begin{aligned}
&\partial_t u + B_1 \partial_{x_1} u + B_2 \partial_{x_2} u + e^{i \frac{\phi_3}{\epsilon}} M u = 0 \text{ in } x_2 > 0 \\
&Bu = \epsilon G(t, x_1, \frac{\phi_0}{\epsilon}) := \epsilon g_{-2}(t, x_1) e^{-i \frac{2\phi_0}{\epsilon}} \text{ on } x_2 = 0 \\
&u = 0 \text{ in } t < 0.
\end{aligned} \tag{2.35}$$

Here the B_j and M are constant 3×3 matrices, the B_j are real, B_2 is invertible, and $\phi_0(t, x_1) = \beta_l \cdot (t, x_1)$, where $\beta_l = (\sigma_l, \eta_l) \in \Upsilon_0$. The 2×3 matrix B is real and of rank 2.

The system has characteristic phases $\phi_m(t, x) = \beta_l \cdot (t, x_1) + \omega_m(\beta_l) x_2$, $m = 1, 2, 3$, where ϕ_2, ϕ_3 are incoming and ϕ_1 outgoing. The system is chosen so that $\Upsilon_0 = \{\beta_l, -\beta_l\}$ and so that the *only* resonance is

$$-2\phi_2 + \phi_3 = -\phi_1 \Leftrightarrow -2\omega_2(\beta_l) + \omega_3(\beta_l) = -\omega_1(\beta_l). \tag{2.36}$$

We construct and justify high order approximate solutions to (2.35) exhibiting double amplification of the form

$$u_a^\epsilon(t, x) = \sum_{k=-1}^{J} \epsilon^k U_k \left(t, x, \frac{\Phi}{\epsilon} \right), \tag{2.37}$$

where $\Phi = (\phi_1, \phi_2, \phi_3)$ and the profiles $U_k(t, x, \theta)$ are 2π-periodic with respect to $\theta = (\theta_1, \theta_2, \theta_3)$.[29]

[29] In stating Example 2.14 we use θ as a placeholder for $\frac{\Phi}{\epsilon}$ and θ_0 as a placeholder for $\frac{\phi_0}{\epsilon}$.

Example 2.14 (Instantaneous double amplification). *A system of the form* (2.35) *can be constructed satisfying assumptions 2.1, 2.3, 2.9 with* $\Upsilon_0 = \{\beta_l, -\beta_l\}$*, where the phases* ϕ_j*,* $j = 1, 2, 3$ *exhibit just one resonance, namely* (2.36)*. For any* $T > 0$ *let* $\Omega_T = (-\infty, T] \times \{(x_1, x_2) : x_2 \geq 0\}$ *and consider the problem* (2.35) *on* Ω_T*, where* $g_{-2}(t, x_1) \in H^\infty((-\infty, T] \times \mathbb{R})$ *and vanishes in* $t < 0$*. For an appropriate choice of the matrix* M *the following conclusions hold:*

a) The problem has a unique exact solution $u^\epsilon \in H^\infty(\Omega_T)$*. Moreover,* $u^\epsilon = U^\epsilon(t, x, \theta_0)|_{\theta_0 = \frac{\phi_0}{\epsilon}}$*, where* $U^\epsilon(t, x, \theta_0)$ *is the solution given by Theorem 2.11(b) to the singular problem* (1.7) *corresponding to* (2.35)*.*

b) For any $Q \in \mathbb{N}$*, the problem has a high order approximate solution* $u^\epsilon_a(t, x) \in H^\infty(\Omega_T)$ *of the form* (2.37) *with* $J = J(Q)$*, which satisfies*

$$|u^\epsilon(t, x) - u^\epsilon_a(t, x)|_{L^\infty(\Omega_T)} = O(\epsilon^Q). \tag{2.38}$$

c) The leading profile $U_{-1}(t, x, \theta)$ *is generally nonzero (and independent of* ϵ*) for arbitrarily small* $t > 0$*.*[30] *Thus, the exact solution* u^ϵ *exhibits instantaneous double amplification, that is, amplification by a factor of* $\frac{1}{\epsilon^2}$ *relative to the boundary data.*[31]

Details on how to construct an explicit system of the form (2.35) satisfying the conclusions of Example 2.14 are given in Remark 5.2.

3 Iteration estimate

In this section we prove the iteration estimate (Proposition 3.18) after presenting some definitions and tools needed for its statement and proof. Our first task is to choose suitable extensions to Ξ of the functions $R_j(\zeta)$, $\omega_j(\zeta)$, $r_j(\zeta)$, $\Delta_a(\zeta)$, and $\Delta(\zeta)$ defined in section 2.

3.1 Extensions to $\Gamma_\delta(\beta)$ and then to Ξ.

Now fix $\beta \in \Upsilon_0$ and a conic neighborhood $\Gamma^+_\delta(\beta) \ni \beta$ as before. We first extend the vectors R_j from $\Gamma^+_\delta(\beta)$ to the conic set $\Gamma_\delta(\beta) = \Gamma^+_\delta(\beta) \cup \Gamma^-_\delta(\beta) \subset \Xi$, where

$$\Gamma^-_\delta(\beta) := \{(\sigma - i\gamma, \eta) : (-\sigma - i\gamma, -\eta) \in \Gamma^+_\delta(\beta)\}, \tag{3.1}$$

by setting

$$R_j(\sigma - i\gamma, \eta) = \overline{R}_j(-\sigma - i\gamma, -\eta) \text{ for } (\sigma - i\gamma, \eta) \in \Gamma^-_\delta(\beta). \tag{3.2}$$

The extended R_j are analytic in τ, C^∞ in η, and homogeneous of degree 1 in $\Gamma_\delta(\beta)$. Next we take C^∞ extensions of these vectors to the rest of Ξ with the property that $S(\zeta)$, defined as in (2.13) but using the extended R_j to define $r_j = R_j/|R_j|$, is invertible with a uniformly bounded inverse $S^{-1}(\zeta)$ on Ξ. As before we denote the rows of $S^{-1}(\zeta)$ by $\ell_j(\zeta)$, $j = 1, \ldots, N$.

We extend the eigenvalues $\omega_j(\zeta)$ from $\Gamma^+_\delta(\beta)$ to $\Gamma_\delta(\beta)$ in essentially the same way:

$$\omega_j(\sigma - i\gamma, \eta) = -\overline{\omega}_j(-\sigma - i\gamma, -\eta) \text{ for } (\sigma - i\gamma, \eta) \in \Gamma^-_\delta(\beta). \tag{3.3}$$

[30] The proof of the conclusions in Example 2.14 will clarify the sense in which "generally" here means "for most choices of M and g_{-2}" in (2.35).

[31] The problem is linear, so an analogous result holds when the factor ϵ on $g_{-2}(t, x_1)$ in (2.35) is replaced by any other power of ϵ.

We then further extend the ω_j to the rest of Ξ as C^∞ functions homogeneous of degree 1.

As with R_j, each ω_j is analytic in τ (and smooth in η) on $\Gamma_\delta(\beta)$. Moreover, since the matrices A_j in (2.6) are real, we have

$$\mathcal{A}(\zeta)R_j(\zeta) = \omega_j(\zeta)R_j(\zeta) \text{ on } \Gamma_\delta(\beta). \tag{3.4}$$

It follows directly from (3.3) that we now have on $\Gamma_\delta(\beta)$:

$$\begin{aligned} &\operatorname{Im} \omega_j(\zeta) \leq -c\gamma \text{ for } j \in \mathcal{O} \\ &\operatorname{Im} \omega_j(\zeta) \geq c\gamma \text{ for } j \in \mathcal{I}, \end{aligned} \tag{3.5}$$

for c as in (2.11). Thus, we see that the extended $R_j(\zeta) \in \mathbb{E}^s(\zeta)$ for $j \in \mathcal{I}$ on $\Gamma_\delta(\beta)$. Noting also that since the $R_j(\beta)$ are real we have $\det BR_-(-\beta) = 0$, we obtain:[32]

Proposition 3.1. *Under Assumptions 2.1, 2.3, and 2.9, if $\beta \in \Upsilon_0$ then $-\beta \in \Upsilon_0$. Thus, $|\Upsilon_0| = M_0$ is even, and there exist vectors $\beta_1, \dots, \beta_{\frac{M_0}{2}}$ in Σ_0 such that*

$$\Upsilon_0 = \{\pm\beta_1, \dots, \pm\beta_{\frac{M_0}{2}}\}. \tag{3.6}$$

The extensions of the R_j similarly give us extensions of Δ_a and Δ to Ξ, with Δ_a (resp., Δ) having the same regularity as the R_j (resp., as the r_j). The extension $\Delta(\zeta)$ clearly satisfies (recall (2.21))

$$|\Delta(\zeta)| \sim \frac{|\tau - c_+\eta|}{|\zeta|} \text{ on } \Gamma_\delta(\beta), \tag{3.7}$$

and we choose the extensions of the R_j so that

$$|\Delta(\zeta)| \sim 1 \text{ on } \Xi \setminus \Gamma_\delta(\beta). \tag{3.8}$$

Remark 3.2. *1) The constant c_+ in* (2.20), (3.7) *depends on the choice of $\beta \in \Upsilon_0$, so we will* sometimes *write $c_+(\beta)$ to indicate this.*

2) The fully extended functions ω_j, $R_j(\zeta)$, $r_j(\zeta)$, $\Delta(\zeta)$, $S(\zeta)$ are defined on Ξ, but were first defined on $\Gamma_\delta(\beta)$, so they all depend on the choice of $\beta \in \Upsilon_0$. Thus, we will sometimes write $\omega_j(\zeta;\beta)$, $R(\zeta;\beta)$, $\Delta(\zeta;\beta)$, etc., when it is important to recall this. Observe that generally $\omega_j(\zeta;\beta_k) \neq \omega_j(\zeta;\beta_m)$ if $k \neq m$.

3) It follows from our construction that there exists a constant C independent of $\zeta \in \Xi$ and $j \in \{1, \dots, M_0\}$ such that

$$|S(\zeta;\beta_j)| \leq C \text{ and } |S^{-1}(\zeta;\beta_j)| \leq C \text{ for all } \zeta \in \Xi,\ j \in \{1, \dots, M_0\}. \tag{3.9}$$

3)It will be convenient to set $\Upsilon_0^+ = \{\beta_1, \dots, \beta_{\frac{M_0}{2}}\}$, where these $\beta_j = (\sigma_j, \eta_j)$ are the (possibly relabeled) elements of Υ_0 such that $\sigma_j \geq 0$. In the ambiguous case where $\sigma_j = 0$ we take $\beta_j = (0,1)$. It is no restriction to (and we do now) suppose that β_l (as in (2.2)*) is in Υ_0^+. Moreover, we shall henceforth always take the β that appears in expressions like $R(\zeta;\beta)$, $\Delta(\zeta;\beta)$, etc., to be an element of Υ_0^+.*

The following proposition is a simple consequence of (3.7).

[32] The above extension of the ω_j and R_j from $\Gamma_\delta^+(\beta)$ to $\Gamma_\delta(\beta)$ satisfying (3.4) and (3.5) works near any $\beta \in \mathcal{H}$, not just $\beta \in \Upsilon_0$.

Proposition 3.3. *The $p \times p$ matrix $Br_-(\zeta)$ satisfies*

$$\left|[Br_-(\zeta)]^{-1}\right| \lesssim |\Delta(\zeta)|^{-1} \sim \frac{|\zeta|}{|\tau - c_+(\beta)\eta|} \text{ on } \Gamma_\delta(\beta). \tag{3.10}$$

Notations 3.4. *1. For $\zeta = (\tau, \eta) \in \Xi$ we set $X_k := \zeta + \frac{k\beta_l}{\epsilon}$ for $\beta_l \in \Upsilon_0^+$ as in (1.13).*[33]

2. Given any function $f(\zeta)$ defined for $\zeta \in \Xi$, we denote by $f(\epsilon, k)$ (with slight abuse) the function of ζ given by $f(X_k)$ for that particular choice of (ϵ, k). This defines the functions $r_\pm(\epsilon, k; \beta)$, $\ell_\pm(\epsilon, k; \beta)$, $S(\epsilon, k; \beta)$, $\Delta(\epsilon, k; \beta)$, etc. that we use below. These functions change, of course, as β varies. Observe, for example, that [34]

$$\omega_j(\epsilon, k; \beta)(\zeta) = \omega_j(X_k; \beta). \tag{3.11}$$

3. Let $\chi_b(\zeta; \beta)$ be the characteristic function of $\Gamma_\delta(\beta)$. Thus, $\zeta \in \operatorname{supp} \chi_b(\epsilon, k; \beta)$ if and only if $X_k \in \Gamma_\delta(\beta)$.

3.2 Tools for the iteration estimate

In the proof of the iteration estimate we will repeatedly use the following simple proposition, where $|f|_{L^2(x_2,\sigma,\eta)}$ denotes the L^2 norm over $\mathbb{R}^+_{x_2} \times \mathbb{R}_\sigma \times \mathbb{R}_\eta$. The proof, given in [Wil20], is almost immediate.

Proposition 3.5. *For $\gamma > 0$, $\tau = \sigma - i\gamma$ we have on $x_2 \geq 0$:*

$$\begin{aligned}
&(a) \left|\int_0^{x_2} e^{-\gamma(x_2 - s)} f(s, \tau, \eta) ds\right|_{L^2(x_2,\sigma,\eta)} \leq \frac{1}{\gamma}|f|_{L^2(x_2,\sigma,\eta)} \\
&(b) \left|\int_{x_2}^{\infty} e^{\gamma(x_2 - s)} f(s, \tau, \eta) ds\right|_{L^2(x_2,\sigma,\eta)} \leq \frac{1}{\gamma}|f|_{L^2(x_2,\sigma,\eta)} \\
&(c) \left|\int_0^{\infty} e^{-\gamma s} f(s, \tau, \eta) ds\right|_{L^2(\sigma,\eta)} \leq \frac{1}{\sqrt{2\gamma}}|f|_{L^2(x_2,\sigma,\eta)} \\
&(d) \left|e^{-\gamma x_2} g(\tau, \eta)\right|_{L^2(x_2,\sigma,\eta)} = \frac{1}{\sqrt{2\gamma}}|g|_{L^2(\sigma,\eta)}.
\end{aligned} \tag{3.12}$$

We turn now to the problem of estimating $V_k(x_2, \zeta)$ as in (1.13). For X_k outside a conic neighborhood of the set of bad directions Υ_0 we can expect to have a Kreiss-type estimate. The main new problem is to estimate V_k for X_k in a neighborhood $\Gamma_\delta(\beta)$ for any given $\beta \in \Upsilon_0^+$; for such X_k the quantity $|\Delta(X_k; \beta)|^{-1}$ can blow up as $\epsilon \to 0$.

The next lemma is an easy consequence of the estimates on $\Delta(\zeta; \beta)$ in section 3.1 and definitions given there.

[33] Unlike β, the frequency β_l appearing in (1.13) and (2.2) is fixed once and for all.

[34] This notation is intended to emphasize that we are viewing the function in (3.11) as a function of ζ for fixed (ϵ, k).

Lemma 3.6. *Fix $\beta \in \Upsilon_0^+$. Recall that $X_k := \zeta + k\frac{\beta_l}{\epsilon}$ and that $\zeta \in \text{supp}\ \chi_b(\epsilon,k;\beta) \Leftrightarrow X_k \in \Gamma_\delta(\beta)$. For $k \in \mathbb{Z}$ the following estimates hold:*[35]

$$
\begin{aligned}
&(a)\ |\Delta(\epsilon,k)| \lesssim 1 \ \text{on}\ \Xi\\
&(b)\ \left|[Br_-(\epsilon,k)]^{-1}\right| \lesssim |\Delta(\epsilon,k)|^{-1} \sim \frac{|X_k|}{|(\tau+\frac{k\sigma_l}{\epsilon})-c_+(\beta)(\eta+\frac{k\eta_l}{\epsilon})|} \ \text{on supp}\ \chi_b(\epsilon,k;\beta)\\
&(c)\ \left|[Br_-(\epsilon,k)]^{-1}\right| \lesssim |\Delta(\epsilon,k)|^{-1} \le C(\delta) \ \text{on}\ \Xi\setminus \text{supp}\ \chi_b(\epsilon,k;\beta)\\
&(d)\ |Br_\pm(\epsilon,k)| \lesssim 1 \ \text{on}\ \Xi,\\
&(e)\ |\omega_i(\epsilon,k)-\omega_j(\epsilon,k)| \sim |X_k| \ \text{for}\ i\neq j \ \text{on supp}\ \chi_b(\epsilon,k;\beta)\\
&(f)\ \text{Im}\,\omega_j(\epsilon,k) \le -c\gamma \ \text{for}\ j\in\mathcal{O}, \quad \text{Im}\,\omega_j(\epsilon,k)\ge c\gamma \ \text{for}\ j\in\mathcal{I}, \ \text{on supp}\ \chi_b(\epsilon,k;\beta).
\end{aligned}
\tag{3.13}
$$

(g) Let $r \in \mathbb{Z}\setminus 0$. When $X_k \notin \Gamma_{\frac{\delta}{|r|}}(\beta)$ we have $|\Delta(\epsilon,k)|^{-1} \le C(\delta)|r|$.

Remark 3.7. *When $\beta = \beta_l$ the denominator in* (3.13)*(b) is simply $|\tau - c_+(\beta_l)\eta|$.*

The functions $E_{i,j}(\epsilon,k,k-r;\beta)$ in the next definition arise as exponents in the oscillatory integrals of step **7** of the iteration estimate (Proposition 3.18). They are used to control the large factors $[Br_-(\epsilon,k)]^{-1}$ appearing in (3.39). Recall that for a given (ζ,ϵ), we have $\omega_i(\epsilon,k;\beta)(\zeta) = \omega_i(\zeta;\beta)|_{\zeta=X_k}$, and, moreover, $\omega_N(\beta_l) = \omega_N(\zeta;\beta_l)|_{\zeta=\beta_l}$.

Definition 3.8. *Let $k \in \mathbb{Z}$, $r \in \mathbb{Z}\setminus 0$. For $\zeta = (\tau,\eta)\in\Xi$ we define the function of ζ:*

$$
E_{i,j}(\epsilon,k,k-r;\beta) := \omega_i(\epsilon,k;\beta) - \frac{r\omega_N(\beta_l)}{\epsilon} - \omega_j(\epsilon,k-r;\beta), \ \text{where}\ i\in\mathcal{O},\ j\in\mathcal{I}
\tag{3.14}
$$

The next two propositions, which are proved in section 3.5, address important technical issues that arise in the proof of the iteration and cascade estimates. In particular, they are needed for defining and controlling the amplification factors $\mathbb{D}(\epsilon,k,k-r)$. In these propositions we fix an *admissible sequence* of integers.

Definition 3.9. *The sequence $(k_j)_{j\in\mathbb{Z}}$ is* admissible *if $k_j \in \mathbb{Z}$ for all j and (k_j) is either strictly increasing or strictly decreasing. We define the "step size" $r_j = k_j - k_{j-1} \in \mathbb{Z}\setminus 0$.*

Proposition 3.10. *Let $\beta \in \Upsilon_0^+$ and suppose $i\in\mathcal{O}$, $j\in\mathcal{I}$, There exist positive constants ϵ_0, δ_0 and positive constants C_3, C_4* independent of $(\zeta,\epsilon,p)\in\Xi\times(0,\epsilon_0]\times\mathbb{Z}$ *such that the following situation holds:*

1) If $\beta\neq\beta_l$, then for any given $(\zeta,\epsilon,\delta)\in\Xi\times(0,\epsilon_0]\times(0,\delta_0]$, either at least one of $X_{k_p}, X_{k_{p-1}}$ does not lie in $\in\Gamma_{\frac{\delta}{|r_p|}}(\beta)$, or both points lie in $\in\Gamma_{\frac{\delta}{|r_p|}}(\beta)$ and

$$
|E_{i,j}(\epsilon,k_p,k_{p-1};\beta)| \ge C_3|X_{k_{p-1}}|.
\tag{3.15}
$$

Moreover, in the second case we have $|X_{k_{p-1}}| \gtrsim \frac{|X_{k_p}|}{|r_p|}$.

2) Suppose $\beta=\beta_l$, and for any given $(\zeta,\epsilon,\delta)\in\Xi\times(0,\epsilon_0]\times(0,\delta_0]$ let the set $M_{i,j}(\zeta,\epsilon,\delta;C_3)\subset\mathbb{Z}$ be characterized by the condition that $p\notin M_{i,j}(\zeta,\epsilon,\delta;C_3)$ if and only if either

[35] Here and below we use Notation 3.4, and often suppress β and ζ dependence; thus, for example, we write $\Delta(\epsilon,k) = \Delta(\epsilon,k;\beta)$ when β clear from the context. In (3.13)(b), for example, $\Delta(\epsilon,k)$ is evaluated at the same ζ that appears in the definition of X_k.

a) at least one of $X_{k_p}, X_{k_{p-1}}$ does not lie in $\in \Gamma_{\frac{\delta}{|r_p|}}(\beta_l)$, or

b) both points lie in $\Gamma_{\frac{\delta}{|r_p|}}(\beta_l)$ and

$$|E_{i,j}(\epsilon, k_p, k_{p-1}; \beta_l)| \geq C_3 \frac{|X_{k_p}|}{|r_p|} \text{ or } |E_{i,j}(\epsilon, k_p, k_{p-1}; \beta_l)| \geq C_3 |X_{k_{p-1}}|. \tag{3.16}$$

Then for every $p \in M_{i,j}(\zeta, \epsilon, \delta; C_3)$ we have

$$|X_{k_p}| \leq \frac{C_4 |r_p|}{\epsilon}. \tag{3.17}$$

Thus, $M_{i,j}(\zeta, \epsilon, \delta; C_3)$ is a set of "bad" p values: p lies in this set if and only if both $X_{k_p}, X_{k_{p-1}}$ lie in $\in \Gamma_{\frac{\delta}{|r_p|}}(\beta_l)$ and (3.16) fails to hold. We are able to prove energy estimates like (2.28) only in those cases where, for any given (i, j, δ) as above, the set $M_{i,j}(\zeta, \epsilon, \delta; C_3)$ has a finite cardinality bounded above by numbers $\mathbb{M}_{i,j}$ *independent of* $(\zeta, \epsilon, \delta)$ and the choice of sequence (k_p); see Proposition 4.2. The exponent $\mathbb{E}$ in (2.28) is determined in part by the size of these upper bounds.

Proposition 3.11. *Let $(k_j)_{j \in \mathbb{Z}}$ be an admissible sequence, and fix $\beta \in \Upsilon_0^+$, $\beta \neq \beta_l$. Consider any given $(\zeta, \epsilon) \in \Xi \times (0, \epsilon_0]$. Provided δ and ϵ_0 are small enough, there exist positive constants C_1, C_2 independent of (ζ, ϵ, j) such that for all but at most two $j \in \mathbb{Z}$ we have:*

$$\left| \frac{\Delta(\epsilon, k_{j-1})}{\Delta(\epsilon, k_j)} \right| \leq C_1 |r_j|^2. \tag{3.18}$$

If an exceptional case occurs (that is, if (3.18) *fails for some $j = m(\zeta, \epsilon; \beta)$), we have*

$$\left| \frac{\Delta(\epsilon, k_{m-1})}{\Delta(\epsilon, k_m)} \right| \leq \frac{C_2 |r_m|^2}{\epsilon \gamma}. \tag{3.19}$$

Remark 3.12 (Uniformity with respect to sequences $(k_j)_{j \in \mathbb{Z}}$). *(1) The previous two propositions were stated with respect to a given admissible sequence $(k_j)_{j \in \mathbb{Z}}$. The statements involve certain constants: ϵ_0, δ_0 and C_i, $i = 1, \dots, 4$. It is important for our application to the cascade estimates that these constants can be chosen independently of the sequence $(k_j)_{j \in \mathbb{Z}}$. Indeed, the proofs show that dependence on this sequence occurs only in the explicit step-size factors r_j that appear in estimates like* (3.18) *or* (3.16).

(2) We will sometimes apply Propositions 3.11 and 3.10 to a fixed pair k, $k - r$ instead of to a given admissible sequence (k_j). To do this we can imagine the pair $k - r, k$ as embedded in a strictly increasing sequence as $k - r = k_{j-1}$, $k = k_j$ with $r = r_j$ if $r > 0$, and in a strictly decreasing sequence if $r < 0$. Part (1) of this remark shows that the constants $C_1, \dots, C_4$ do not depend on the admissible sequence chosen for the embedding. In the same way we can apply Proposition 3.21 (a refinement of Proposition 3.10 proved in section 3.5) to fixed pairs k, $k - r$.

(3) Although admissible sequences are needed only for the one-sided cascade estimates (for example, in Proposition 4.3), part (2) of this remark allows us to use the above two propositions in the two-sided case as well.

The following lemma is used in defining amplication factors and in the proof of the iteration estimate.

Lemma 3.13. *Let $X_k = \zeta + k\frac{\beta_l}{\epsilon}$, $X_{k-r} = \zeta + (k-r)\frac{\beta_l}{\epsilon}$, where $r \in \mathbb{Z} \setminus 0$. For $\delta \in (0, \delta_0]$ and $N_1 \in \mathbb{N}$ sufficiently large, assume that*

$$X_k \in \Gamma_{\frac{\delta}{N_1|r|}}(\beta_l), \text{ but } X_{k-r} \notin \Gamma_{\frac{\delta}{|r|}}(\beta_l). \tag{3.20}$$

Then $|X_{k-r}| \lesssim \frac{1}{N_1}|X_k|$.

Proof. **1.** Let $\tilde{X}_{k-r} = t\frac{\beta_l}{\epsilon}$ and $\tilde{X}_k = (t+r)\frac{\beta_l}{\epsilon}$ be the orthogonal projections of X_{k-r} and X_k on the β_l axis, and let ℓ denote the common distance of X_{k-r} and X_k from that axis. By (3.20) we have

$$\frac{\ell}{|\tilde{X}_k|} \lesssim \frac{\delta}{N_1|r|}, \quad \frac{\ell}{|\tilde{X}_{k-r}|} \gtrsim \frac{\delta}{|r|}, \tag{3.21}$$

and thus

$$\frac{\delta}{|r|}\frac{|t|}{\epsilon} \lesssim \ell \lesssim \frac{\delta}{N_1|r|}\frac{|t+r|}{\epsilon}, \text{ which implies } \frac{|t+r|}{N_1} \gtrsim |t|; \text{ hence } |t| \lesssim \frac{|r|}{N_1 - 1}. \tag{3.22}$$

2. Now

$$\ell \sim \frac{\delta}{N_1|r|}|\tilde{X}_k| \sim \frac{\delta}{N_1|r|}\frac{|t+r|}{\epsilon} \lesssim \frac{\delta}{N_1|r|\epsilon}\left(\frac{|r|}{N_1-1} + |r|\right) \lesssim \frac{\delta}{N_1\epsilon}, \text{ so} \tag{3.23}$$

$$|X_{k-r}| \lesssim \ell + \frac{|t|}{\epsilon} \lesssim \frac{|r|}{N_1\epsilon}, \text{ while } |X_k| \sim |\tilde{X}_k| \sim \frac{|t+r|}{\epsilon} \sim \frac{|r|}{\epsilon}, \tag{3.24}$$

establishing the lemma.

□

3.3 Amplification factors

In this section we define the global amplification factors $\mathbb{D}(\epsilon, k, k-r)$ that appear in the iteration estimate (3.18). Each global factor $\mathbb{D}(\epsilon, k, k-r)$ is constructed out of microlocal factors $D(\epsilon, k, k-r; \beta)$ that we now proceed to define. In this discussion the constants ϵ_0, δ_0, C_i, $i = 1, \ldots, 4$, are those from Propositions 3.11 and 3.10. We take $(\zeta, \epsilon, \delta, k, r) \in \Xi \times (0, \epsilon_0] \times (0, \delta_0] \times \mathbb{Z} \times (\mathbb{Z} \setminus 0)$ and $\beta \in \Upsilon_0^+$.

First, for δ and $N_1 \geq 2$ as in Lemma 3.13 we distinguish three cases for the pair $X_k = \zeta + k\frac{\beta_l}{\epsilon}$, $X_{k-r} = \zeta + (k-r)\frac{\beta_l}{\epsilon}$:

$$\begin{aligned} &(I)\ X_k \in \Gamma_{\frac{\delta}{N_1|r|}}(\beta), X_{k-r} \in \Gamma_{\frac{\delta}{|r|}}(\beta) \\ &(II)\ X_k \in \Gamma_{\frac{\delta}{N_1|r|}}(\beta), X_{k-r} \notin \Gamma_{\frac{\delta}{|r|}}(\beta) \\ &(III)\ X_k \in \Gamma_\delta(\beta) \setminus \Gamma_{\frac{\delta}{N_1|r|}}(\beta) \end{aligned} \tag{3.25}$$

The source of amplification is the factor $[Br_-(\epsilon, k)]^{-1}$ that appears (in two places) in (3.39). From Lemma 3.6 we have

$$\begin{aligned} &(a)\left|[Br_-(\epsilon, k)]^{-1}\right| \lesssim |\Delta(\epsilon, k)|^{-1} \lesssim \frac{|X_k|}{\gamma} \text{ for } X_k \in \Gamma_\delta(\beta) \\ &(b)|\Delta(\epsilon, k)|^{-1} \leq C(\delta)|r| \text{ for } X_k \notin \Gamma_{\frac{\delta}{|r|}}(\beta). \end{aligned} \tag{3.26}$$

Using (3.26)(b) we obtain

$$(3.27)\qquad |\Delta(\epsilon,k)|^{-1}\leq C(\delta)N_1|r| \text{ in case } (III).$$

In case (II) we have using Proposition 3.11 and Remark 3.12:

$$(3.28)\qquad \frac{1}{|\Delta(\epsilon,k)|}=\frac{1}{|\Delta(\epsilon,k-r)|}\frac{|\Delta(\epsilon,k-r)|}{|\Delta(\epsilon,k)|}\leq\begin{cases}(C(\delta)|r|)(C_1r^2)=C(\delta)|r|^3 \text{ if (3.18) holds}\\ (C(\delta)|r|)(\frac{C_2r^2}{\epsilon\gamma})=\frac{C(\delta)|r|^3}{\epsilon\gamma} \text{ if (3.18) fails}\end{cases}.$$

In case (I) we will use the functions $E_{i,j}(\epsilon,k,k-r;\beta)$ (and other similar functions) to control $|\Delta(\epsilon,k)|^{-1}$. When $\beta=\beta_l$, for each $(i,j)\in\mathcal{O}\times\mathcal{I}$ we define case (Ia) to be the subcase of case (I) where (3.16) holds and (Ib) the subcase where (3.16) fails.[36] When $\beta\neq\beta_l$ we will also used the following remark, which is proved in step **3** of the proof of Proposition 3.11.

Remark 3.14. *Suppose $\beta\neq\beta_l$. In case I for a given (ζ,ϵ) and any $r\in\mathbb{Z}\setminus 0$, we have $\frac{|\Delta(\epsilon,k-r)|}{|\Delta(\epsilon,k)|}\lesssim r^2$ for all but at most one $k\in\mathbb{Z}$. In the exceptional case we have $\frac{|\Delta(\epsilon,k-r)|}{|\Delta(\epsilon,k)|}\lesssim\frac{r^2}{\epsilon\gamma}$.*

Definition 3.15. *[Microlocal amplification factors] Let $C_5\geq 1$ be a constant depending on $C(\delta)$ as in (3.27), (3.28), N_1 as in (3.25), and the constants $C_1,\dots,C_4$ appearing in Propositions 3.11 and 3.10.*[37]

For $k\in\mathbb{Z}$, $r\in\mathbb{Z}\setminus 0$, $\beta\in\Upsilon_0^+$, and $(\zeta,\epsilon)\in\operatorname{supp}\chi_b(\epsilon,k;\beta)\times(0,\epsilon_0]$, we define:

- $D(\epsilon,k,k-r;\beta)(\zeta)=C_5|r|$ *in case (III)*
- *If $\beta\neq\beta_l$, then* $D(\epsilon,k,k-r;\beta)(\zeta)=\begin{cases}C_5|r|^3 \textit{ in case } (II) \textit{ when } (3.18) \textit{ holds}\\ \frac{C_5|r|^3}{\epsilon\gamma} \textit{ in case } (II) \textit{ when } (3.18) \textit{ fails}\end{cases}$
- $D(\epsilon,k,k-r;\beta_l)(\zeta)=C_5|r|$ *in case (II).*
- *If $\beta\neq\beta_l$,* $D(\epsilon,k,k-r;\beta)(\zeta)=\begin{cases}C_5r^2 \textit{ in case } (I) \textit{ when } (3.18) \textit{ holds}\\ \frac{C_5r^2}{\epsilon\gamma} \textit{ in case } (I) \textit{ when } (3.18) \textit{ fails}\end{cases}$

If $\beta=\beta_l$ and case (I) obtains, then

- $D(\epsilon,k,k-r;\beta_l)(\zeta)=\begin{cases}C_5|r| \textit{ when for every } (i,j)\in\mathcal{O}\times\mathcal{I}, \textit{ case } (Ia) \textit{ holds}\\ \frac{C_5|r|}{\epsilon\gamma} \textit{ when for some } (i,j)\in\mathcal{O}\times\mathcal{I}, \textit{ case } (Ib) \textit{ holds}\end{cases}.$

For any β if $\zeta\notin\operatorname{supp}\chi_b(\epsilon,k,\beta)$, define $D(\epsilon,k,k-r;\beta)(\zeta)=0$.

Definition 3.16. *[Global amplification factors] Let $\chi_{\mathcal{B}}(\zeta)$ be the characteristic function of $\mathcal{B}:=\cup_{\beta\in\Upsilon_0^+}\Gamma_\delta(\beta)$. For $k\in\mathbb{Z}$, $r\in\mathbb{Z}\setminus 0$, and $(\zeta,\epsilon)\in\operatorname{supp}\chi_{\mathcal{B}}(\epsilon,k)\times(0,\epsilon_0]$, we define*

$$(3.29)\qquad \mathbb{D}(\epsilon,k,k-r)(\zeta)=\begin{cases}\frac{C_5|r|^3}{\epsilon\gamma}, \textit{ if } D(\epsilon,k,k-r;\beta)(\zeta)=\frac{C_5|r|}{\epsilon\gamma},\frac{C_5r^2}{\epsilon\gamma} \textit{ or } \frac{C_5|r|^3}{\epsilon\gamma} \textit{ for some } \beta\in\Upsilon_0^+\\ C_5|r|^3, \textit{ otherwise}\end{cases}.$$

If $\zeta\notin\operatorname{supp}\chi_{\mathcal{B}}(\epsilon,k)$ we set $\mathbb{D}(\epsilon,k,k-r)(\zeta)=1$.

Observe that

$$(3.30)\qquad D(\epsilon,k,k-r;\beta)(\zeta)\leq\mathbb{D}(\epsilon,k,k-r) \text{ for all } \zeta\in\Xi,\ \beta\in\Upsilon_0^+.$$

Remark 3.17. *In any problem where $\Upsilon_0^+=\{\beta_l\}$, we replace $|r|^3$ by $|r|$ on the right in (3.29). This remark is used in the proof of Theorem 2.12.*

[36]Part 1 of Proposition 3.10 shows that when $\beta\neq\beta_l$, the estimate (3.15) always holds in case I.

[37]The choice of C_5 is further clarified in the proof of Proposition 3.18.

3.4 Iteration estimate

In this section we prove the iteration estimate for the transformed singular problem (1.13), that is,

$$\begin{aligned}
&(a) D_{x_2}V_k - \mathcal{A}(X_k)V_k = i\sum_{r\in\mathbb{Z}\setminus 0}\alpha_r e^{ir\frac{\omega_N(\beta_l)}{\epsilon}x_2}B_2^{-1}MV_{k-r} + \widehat{F_k^\epsilon}(x_2,\zeta)\\
&(b) BV_k = \widehat{G_k}(\zeta) \text{ on } x_2 = 0.
\end{aligned}\tag{3.31}$$

This is the transform of the singular problem (1.7) in the case where we take

$$\mathcal{D}(\theta_{in}) = d(\theta_{in})M \text{ for } d(\theta_{in}) = \sum_{r\in\mathbb{Z}\setminus 0}\alpha_r e^{ir\theta_{in}}.\tag{3.32}$$

We justify the reduction to this case in Remark 4.8.

We begin by defining the objects that appear in the estimate. Let us set

$$\mathcal{B} := \cup_{\beta\in\Upsilon_0^+}\Gamma_\delta(\beta)\subset\Xi.\tag{3.33}$$

Using Notations 3.4 we write the solution V_k of the Fourier-Laplace transformed singular system (3.31) as

$$V_k = \sum_{\beta\in\Upsilon_0^+}\chi_b(\epsilon,k;\beta)V_k + \chi_g(\epsilon,k)V_k,\tag{3.34}$$

where $\chi_g(\zeta)$ is the characteristic function of $\mathcal{B}^c$ (complement in Ξ). Define $w_k = (w_k^+, w_k^-)$ for *all* $\zeta\in\Xi$ by[38]

$$V_k(x_2,\zeta) = S(\epsilon,k;\beta)w_k(x_2,\zeta;\beta),\tag{3.35}$$

and observe that w_k is for $\zeta\in\text{supp }\chi_b(\epsilon,k;\beta)$ a solution of the diagonalized system[39]

$$\begin{aligned}
&D_{x_2}w_k - \begin{pmatrix}\xi_+(\epsilon,k;\beta) & 0\\ 0 & \xi_-(\epsilon,k;\beta)\end{pmatrix}w_k =\\
&\quad i\sum_{r\in\mathbb{Z}\setminus 0}\alpha_r e^{ir\frac{\omega_N(\beta_l)}{\epsilon}x_2}S^{-1}(\epsilon,k;\beta)B_2^{-1}MS(\epsilon,k-r;\beta)w_{k-r} + S^{-1}(\epsilon,k;\beta)\widehat{F_k^\epsilon}(x_2,\zeta),\\
&BS(\epsilon,k;\beta)w_k = \hat{G}_k \text{ on } x_2 = 0.
\end{aligned}\tag{3.36}$$

Here we have set[40]

$$\begin{aligned}
&\xi_+(\epsilon,k;\beta) = \text{diag }(\omega_1(\epsilon,k;\beta),\dots,\omega_{N-p}(\epsilon,k;\beta))\\
&\xi_-(\epsilon,k;\beta) = \text{diag }(\omega_{N-p+1}(\epsilon,k;\beta),\dots,\omega_N(\epsilon,k;\beta)).
\end{aligned}\tag{3.37}$$

Solutions to the diagonalized system in the case $F=0$ are

(3.38)
$$w_k^+(x_2,\zeta) = \sum_{r\in\mathbb{Z}\setminus 0}\int_{x_2}^\infty e^{i\xi_+(\epsilon,k)(x_2-s)+ir\frac{\omega_N(\beta_l)}{\epsilon}s}\alpha_r[a(\epsilon,k,k-r)w_{k-r}^+(s,\zeta) + b(\epsilon,k,k-r)w_{k-r}^-(s,\zeta)]ds,$$

---

[38]It is important now to keep track of the dependence of the various objects on $\beta\in\Upsilon_0^+$.

[39]Recall $\zeta\in\text{supp }\chi_b(\epsilon,k;\beta)$ if and only if $X_k := \zeta + k\frac{\beta_l}{\epsilon}\in\Gamma_\delta(\beta)$.

[40]In the notation used in Remark 3.2, the frequency $\omega_N(\beta_l)$ that appears in (3.36) is $\omega_N(\zeta;\beta_l)|_{\zeta=\beta_l}$.

(3.39)

$$w_k^-(x_2,\zeta) = -\sum_{r\in\mathbb{Z}\setminus 0}\int_0^{x_2} e^{i\xi_-(\epsilon,k)(x_2-s)+ir\frac{\omega_N(\beta_l)}{\epsilon}s}\alpha_r[c(\epsilon,k,k-r)w_{k-r}^+(s,\zeta)+d(\epsilon,k,k-r)w_{k-r}^-(s,\zeta)]ds -$$
$$e^{i\xi_-(\epsilon,k)x_2}[Br_-(\epsilon,k)]^{-1}Br_+(\epsilon,k)\sum_{r\in\mathbb{Z}\setminus 0}\int_0^{\infty} e^{i\xi_+(\epsilon,k)(-s)+ir\frac{\omega_N(\beta_l)}{\epsilon}s}\alpha_r[a(\epsilon,k,k-r)w_{k-r}^+(s,\zeta)+$$
$$b(\epsilon,k,k-r)w_{k-r}^-(s,\zeta)]ds + e^{i\xi_-(\epsilon,k)x_2}[Br_-(\epsilon,k)]^{-1}\hat{G}_k(\zeta).$$

The matrices a,b,c,d in (3.38), (3.39) can be read off from (3.36) and we have

$$|a(\epsilon,k,k-r)|\lesssim 1, |b(\epsilon,k,k-r)|\lesssim 1, |c(\epsilon,k,k-r)|\lesssim 1, |d(\epsilon,k,k-r)|\lesssim 1 \tag{3.40}$$

uniformly with respect to (ϵ,k,r).

We will also need the following expressions for $w_k^\pm$:

$$w_k^+(x_2,\zeta) = \sum_{r\in\mathbb{Z}\setminus 0}\int_{x_2}^{\infty} e^{i\xi_+(\epsilon,k)(x_2-s)+ir\frac{\omega_N(\beta_l)}{\epsilon}s}\alpha_r M^+(\epsilon,k,k-r)V_{k-r}(s,\zeta)ds \tag{3.41}$$

(3.42)

$$w_k^-(x_2,\zeta) = -\sum_{r\in\mathbb{Z}\setminus 0}\int_0^{x_2} e^{i\xi_-(\epsilon,k)(x_2-s)+ir\frac{\omega_N(\beta_l)}{\epsilon}s}\alpha_r M^-(\epsilon,k,k-r)V_{k-r}(s,\zeta)ds -$$
$$e^{i\xi_-(\epsilon,k)x_2}[Br_-(\epsilon,k)]^{-1}Br_+(\epsilon,k)\sum_{r\in\mathbb{Z}\setminus 0}\int_0^{\infty} e^{i\xi_+(\epsilon,k)(-s)+ir\frac{\omega_N(\beta_l)}{\epsilon}s}\alpha_r M^+(\epsilon,k,k-r)V_{k-r}(s,\zeta)ds +$$
$$e^{i\xi_-(\epsilon,k)x_2}[Br_-(\epsilon,k)]^{-1}\hat{G}_k(\zeta),$$

where the matrices $M^\pm$ are defined in the obvious way and satisfy $|M^\pm(\epsilon,k,k-r)|\lesssim 1$.

Next define

$$\mathcal{W}_k(x_2,\zeta;\beta) = (\tilde{w}_k^+(x_2,\zeta;\beta), w_k^-(x_2,\zeta;\beta), \frac{1}{\sqrt{\gamma}}\tilde{w}_k^+(0,\zeta;\beta), \frac{1}{\sqrt{\gamma}}w_k^-(0,\zeta;\beta)), \tag{3.43}$$

where we have set

$$\tilde{w}_k^+(x_2,\zeta;\beta) = \Delta^{-1}(\epsilon,k;\beta)w_k^+(x_2,\zeta;\beta). \tag{3.44}$$

For each k we define a modified L^2 norm of V_k by[41]

$$\|V_k\|_k = \sum_{\beta\in\Upsilon_0^+}|\chi_b(\epsilon,k;\beta)\mathcal{W}_k(x_2,\zeta;\beta)|_{L^2} + \left|\chi_g(\epsilon,k)\left(V_k(x_2,\zeta),\frac{1}{\sqrt{\gamma}}V_k(0,\zeta)\right)\right|_{L^2}, \tag{3.45}$$

This "partial norm" depends on k (through Δ^{-1} and the cutoffs χ_b, χ_g). It should be viewed as a piece of the "full norm" $|(\|V_k\|)|_{\ell^2(k)}$ of $(V_k)_{k\in\mathbb{Z}}$ that is estimated in Proposition 4.5. We usually suppress the outer subscript k of $\|V_k\|_k$. If $f(\zeta)$ is any function of ζ then

$$\|f(\zeta)V_k\|_k := \sum_{\beta\in\Upsilon_0^+}|\chi_b(\epsilon,k;\beta)f(\zeta)\mathcal{W}_k(x_2,\zeta;\beta)|_{L^2} + \left|\chi_g(\epsilon,k)f(\zeta)\left(V_k(x_2,\zeta),\frac{1}{\sqrt{\gamma}}V_k(0,\zeta)\right)\right|_{L^2}. \tag{3.46}$$

[41]In (3.45) the notation $|\cdot|_{L^2}$ means $|\cdot|_{L^2(x_2,\sigma,\eta)}$ for components that depend on x_2 and $|\cdot|_{L^2(\sigma,\eta)}$ for components that do not.

Note that for U^γ as in Theorem 2.11

$$|(\|V_k\|)|_{\ell^2(k)} \gtrsim |U^\gamma|_{L^2(t,x,\theta)} + \left|\frac{U^\gamma(0)}{\sqrt{\gamma}}\right|_{L^2(t,x_1,\theta)}. \tag{3.47}$$

Proposition 3.18 (Iteration estimate)**.** *Under assumptions 2.1, 2.3, 2.9 consider the transformed singular problem* (3.31)*. There exist positive constants C, γ_0 such that for $\gamma \geq \gamma_0$ the solution V_k of* (2.23) *satisfies for $k \in \mathbb{Z}$,*

$$\|V_k\| \leq \frac{C}{\gamma} \sum_{r\in\mathbb{Z}\setminus 0} \sum_{t\in\mathbb{Z}} \|\alpha_r\alpha_t \mathbb{D}(\epsilon,k,k-r)V_{k-r-t}\| + \frac{C}{\gamma^2}\left|\widehat{F_k}|X_k|\right|_{L^2} + \frac{C}{\gamma^{3/2}}\left|\widehat{G_k}|X_k|\right|_{L^2(\sigma,\eta)}. \tag{3.48}$$

Here we redefine α_0 to be 1.[42]

Proof. **1. Kreiss-type estimate.** On the support of $\chi_g(\zeta)$ the problem $(L(\partial), B)$ satisfies the uniform Lopatinski condition, so by using a singular Kreiss symmetrizer as in [Wil02], we obtain the Kreiss-type estimate

$$\left|\chi_g(\epsilon,k)\left(V_k, \frac{1}{\sqrt{\gamma}}V_k(0)\right)\right|_{L^2} \leq C\left(\frac{1}{\gamma}\left(\sum_{r\in\mathbb{Z}\setminus 0}|\alpha_r V_{k-r}|_{L^2} + |\widehat{F_k}|_{L^2}\right) + \frac{1}{\sqrt{\gamma}}|\widehat{G_k}|_{L^2(\sigma,\eta)}\right). \tag{3.49}$$

Observe that the right side of (3.49) is dominated by the right side of (3.48).

2. Strategy. To prove (3.48) it will suffice to show for each $\beta \in \Upsilon_0^+$ that

$$\begin{aligned}&|\chi_b(\epsilon,k;\beta)\mathcal{W}_k|_{L^2} \leq \\ &\frac{C}{\gamma}\sum_{r\in\mathbb{Z}\setminus 0}\sum_{t\in\mathbb{Z}}|\alpha_r\alpha_t D(\epsilon,k,k-r;\beta)\mathcal{W}_{k-r-t}|_{L^2} + \frac{C}{\gamma^2}\left|\widehat{F_k}|X_k|\right|_{L^2} + \frac{C}{\gamma^{3/2}}\left|\widehat{G_k}|X_k|\right|_{L^2(\sigma,\eta)},\end{aligned} \tag{3.50}$$

where the $\mathcal{W}_j$ are all evaluated at $(x_2,\zeta;\beta)$. We can then deduce (3.48) using (3.30) and (3.49).

3. We now fix $\beta \in \Upsilon_0^+$ and proceed to prove (3.50). We will first treat the case $F = 0$, $G = \sum_{k\in\mathbb{Z}} G_k(t,x_1)e^{ik\theta}$. A crude estimate based just on applying Proposition 3.5 to "$\frac{1}{\Delta(\epsilon,k)}$(3.38)" and (3.39) yields[43]

(3.51)

$$(a)|\chi_b(\epsilon,k)\tilde{w}_k^+|_{L^2} \leq \frac{C}{\gamma}\sum_{r\in\mathbb{Z}\setminus 0}\left(\left|\frac{\chi_b(\epsilon,k)\alpha_r w_{k-r}^+}{\Delta(\epsilon,k)}\right|_{L^2} + \left|\frac{\chi_b(\epsilon,k)\alpha_r w_{k-r}^-}{\Delta(\epsilon,k)}\right|_{L^2}\right),$$

$$(b)|\chi_b(\epsilon,k)w_k^-|_{L^2} \leq \frac{C}{\gamma}\sum_{r\in\mathbb{Z}\setminus 0}\left(\left|\frac{\chi_b(\epsilon,k)\alpha_r w_{k-r}^+}{\Delta(\epsilon,k)}\right|_{L^2} + \left|\frac{\chi_b(\epsilon,k)\alpha_r w_{k-r}^-}{\Delta(\epsilon,k)}\right|_{L^2}\right) + \frac{C}{\gamma^{3/2}}\left|\widehat{G_k}|X_k|\right|_{L^2(\sigma,\eta)}.$$

Observe that for each r we have

$$\left|\frac{\chi_b(\epsilon,k)\alpha_r w_{k-r}^+}{\Delta(\epsilon,k)}\right|_{L^2} + \left|\frac{\chi_b(\epsilon,k)\alpha_r w_{k-r}^-}{\Delta(\epsilon,k)}\right|_{L^2} \sim \left|\frac{\chi_b(\epsilon,k)\alpha_r V_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2}, \tag{3.52}$$

[42]In (3.48) $\|V_k\| = \|V_k\|_k$ and $\|\alpha_r\alpha_t\mathbb{D}(\epsilon,k,k-r)V_{k-r-t}\| = \|\alpha_r\alpha_t\mathbb{D}(\epsilon,k,k-r)V_{k-r-t}\|_{k-r-t}$.

[43]In (3.51) the cutoff $\chi_b(\epsilon,k) = \chi_b(\epsilon,k;\beta)$; similarly, we often suppress the ";β" in other functions when a given $\beta \in \Upsilon_0^+$ is fixed by the context.

so we can (and sometimes will) write (3.51) using V_{k-r} in place of (w^+_{k-r}, w^-_{k-r}) on the right. We proceed to improve the estimate (3.51).

4. Decomposition of $\Gamma_\delta(\beta)$; the $\chi^3_b(\epsilon,k,k-r)$ pieces. Considering the three cases listed in (3.25), for a given $r \in \mathbb{Z}\setminus 0$ we write

$$\chi_b(\epsilon,k) = \chi^1_b(\epsilon,k,k-r) + \chi^2_b(\epsilon,k,k-r) + \chi^3_b(\epsilon,k,k-r), \tag{3.53}$$

where $\chi^i_b(\epsilon,k,k-r)$, $i=1,2,3$ are respectively the characteristic functions of

$$\begin{aligned}
&\mathcal{A}_1(\epsilon,k,k-r) := \{\zeta \in \Xi : X_k \in \Gamma_{\frac{\delta}{N_1|r|}}(\beta), X_{k-r} \in \Gamma_{\frac{\delta}{|r|}}(\beta)\}\\
&\mathcal{A}_2(\epsilon,k,k-r) := \{\zeta \in \Xi : X_k \in \Gamma_{\frac{\delta}{N_1|r|}}(\beta), X_{k-r} \notin \Gamma_{\frac{\delta}{|r|}}(\beta)\}\\
&\mathcal{A}_3(\epsilon,k,k-r) := \{\zeta \in \Xi : X_k \in \Gamma_\delta(\beta)\setminus\Gamma_{\frac{\delta}{N_1|r|}}(\beta)\}.
\end{aligned} \tag{3.54}$$

Here $N_1 \in \mathbb{N}$ is chosen as in Proposition 3.13 and $\delta \in (0,\delta_0]$. By Proposition 3.10(1) $\mathcal{A}_1(\epsilon,k,k-r)$ is empty when $\beta \neq \beta_l$.

Since $|\Delta(\epsilon,k)|^{-1} \le C(\delta)N_1|r|$ on supp $\chi^3_b(\epsilon,k,k-r)$, we obtain:

$$\frac{C}{\gamma}\left|\frac{\chi^3_b(\epsilon,k,k-r)w^\pm_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2} \le \frac{C(\delta)N_1|r|}{\gamma}|w^\pm_{k-r}|_{L^2} \le \frac{C}{\gamma}|D(\epsilon,k,k-r;\beta)w^\pm_{k-r}|_{L^2}. \tag{3.55}$$

5. The $\chi^2_b(\epsilon,k,k-r)$ pieces when $\beta \neq \beta_l$. Consider now the piece of the second term on the right of (3.51)(a) given by $\frac{C}{\gamma}\left|\frac{\chi^2_b(\epsilon,k,k-r)\alpha_r w^-_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2}$, which arises from the following part of $\frac{\chi^2_b(\epsilon,k,k-r)}{\Delta(\epsilon,k)}$(3.38):

$$A_2 := \frac{\chi^2_b(\epsilon,k,k-r)}{\Delta(\epsilon,k)}\int_{x_2}^\infty e^{i\xi_+(\epsilon,k)(x_2-s)+i\frac{r\omega_N(\beta_l)}{\epsilon}s}\alpha_r b(\epsilon,k,k-r)w^-_{k-r}(s,\zeta)ds, \tag{3.56}$$

Using (3.28) and Definition 3.15 we obtain

$$\left|\frac{\chi^2_b(\epsilon,k,k-r)}{\Delta(\epsilon,k)}\right| \lesssim D(\epsilon,k,k-r;\beta). \tag{3.57}$$

This yields

$$|A_2|_{L^2} \le \frac{C}{\gamma}|\alpha_r D(\epsilon,k,k-r;\beta)w^-_{k-r}|_{L^2}. \tag{3.58}$$

The $\chi^2_b(\epsilon,k,k-r)$ pieces of the other terms on the right in (3.51) are estimated in the same way.

6. The $\chi^2_b(\epsilon,k,k-r)$ pieces when $\beta = \beta_l$. For $i \in \mathcal{O}$ we define the $N\times N$ matrix

$$E_i(\epsilon,k,k-r) = (\omega_i(X_k) - r\omega_N(\beta_l/\epsilon))\,I_N - \mathcal{A}(X_{k-r}). \tag{3.59}$$

Lemma 3.13 implies that on the support of $\chi^2_b(\epsilon,k,k-r;\beta_l))$, the first two terms on the right of (3.59) dominate the third. More precisely, we have

$$|X_{k-r}| = \left|X_k - \frac{r\beta_l}{\epsilon}\right| \le \frac{1}{N_1}|X_k|, \tag{3.60}$$

so, after enlarging N_1 if necessary, we have[44]

$$|\mathcal{A}(X_{k-r})| \lesssim |X_{k-r}| \leq \frac{1}{N_1}|X_k| \sim \frac{1}{N_1}|\omega_i(X_k) - \omega_N(r\beta_l/\epsilon)|. \tag{3.61}$$

This implies that the $N \times N$ matrix E_i is invertible with

$$|E_i^{-1}| \lesssim \frac{1}{|X_k|}, \text{ and hence } \left|\frac{1}{\Delta(\epsilon,k)}E_i^{-1}(\epsilon,k,k-r)\right| \lesssim \frac{1}{\gamma}, \tag{3.62}$$

since $|\Delta(\epsilon,k)^{-1}| \lesssim |X_k|/\gamma$ on supp $\chi_b^2(\epsilon,k,k-r)$.

For a given r the main contribution to the term (in parentheses) on the right in (3.51)(b) arises from the part of (3.42) given by[45]

$$A := e^{i\xi_-(\epsilon,k)x_2}[Br_-(\epsilon,k)]^{-1}Br_+(\epsilon,k)\int_0^\infty e^{i\xi_+(\epsilon,k)(-s)+i\frac{r\omega_N(\beta_l)}{\epsilon}s}\alpha_r M^+(\epsilon,k,k-r)V_{k-r}(s,\zeta)ds. \tag{3.63}$$

Recalling the definitions of $\xi_\pm$ (3.37) and ignoring some factors in the integrand that are $O(1)$ and independent of s, we see that the $q-$component of A is a sum of terms of the form

$$A_{q,i} := e^{i\omega_q(\epsilon,k)x_2}\frac{1}{\Delta(\epsilon,k)}\int_0^\infty e^{i\omega_i(\epsilon,k)(-s)+i\frac{r\omega_N(\beta_l)}{\epsilon}s}\alpha_r M_i^+(\epsilon,k,k-r)V_{k-r}(s,\zeta)ds \tag{3.64}$$

where $q \in \mathcal{I}$, $i \in \mathcal{O}$, and M_i^+ is the $i-$th row of the $(N-p)\times N$ matrix M^+.

We now improve the estimate of $\chi_b^2(\epsilon,k,k-r)A_{q,i}$ by an integration by parts. Setting

$$V_{k-r}^*(x_2,\zeta) := e^{-i\mathcal{A}(X_{k-r})x_2}V_{k-r}, \tag{3.65}$$

we may rewrite $\chi_b^2(\epsilon,k,k-r)A_{q,i}$ as

$$e^{i\omega_q(\epsilon,k)x_2}\frac{M_i^+(\epsilon,k,k-r)}{\Delta(\epsilon,k)}\int_0^\infty e^{-iE_i(\epsilon,k,k-r)s}\chi_b^2(\epsilon,k,k-r)\alpha_r V_{k-r}^*(s,\zeta)ds, \tag{3.66}$$

where $E_i(\epsilon,k,k-r)$ is given by (3.59). From $(3.31)_{k-r}$ we obtain

$$\partial_{x_2}V_{k-r}^* = e^{-i\mathcal{A}(X_{k-r})x_2}ih_{k-r}, \tag{3.67}$$

where h_{k-r} is the right side of $(3.31)_{k-r}(a)$. Using (3.67), the equation

$$\frac{1}{-i}E_i^{-1}\frac{d}{ds}e^{-iE_is} = e^{-iE_is}, \tag{3.68}$$

and integrating by parts in (3.66) gives

$$\begin{aligned} &e^{i\omega_q(\epsilon,k)x_2}\frac{M_i^+(\epsilon,k,k-r)}{i\Delta(\epsilon,k)}\chi_b^2(\epsilon,k,k-r)E_i^{-1}(\epsilon,k,k-r)\cdot \\ &\left[\alpha_r V_{k-r}(0,\zeta) + \int_0^\infty e^{-iE_i(\epsilon,k,k-r)s}e^{-i\mathcal{A}(X_{k-r})s}\alpha_r ih_{k-r}(s,\zeta)ds\right]. \end{aligned} \tag{3.69}$$

[44] Use (3.60) and (3.13)(e) for the last $\sim$ of (3.61).

[45] The "smaller" contribution from the integrals $\int_0^{x_2}$ in (3.42) (or (3.39)) is easily estimated by direct application of Proposition 3.5, so we ignore it here.

We now obtain from (3.62) and the definition of $\mathcal{D}(\epsilon,k,k-r;\beta_l)$ (in case (II) of (3.25)):

$$
\begin{aligned}
&|\chi_b^2(\epsilon,k,k-r)A_{q,i}|_{L^2}\lesssim \\
&\frac{C}{\gamma^{3/2}}\left[|\alpha_r D(\epsilon,k,k-r;\beta_l)V_{k-r}(0,\zeta)|_{L^2(\sigma,\eta)}+\frac{C}{\sqrt{\gamma}}|\alpha_r D(\epsilon,k,k-r;\beta_l)h_{k-r}|_{L^2}\right]\leq \\
&\frac{C}{\gamma^{3/2}}\left[|\alpha_r D(\epsilon,k,k-r;\beta_l)V_{k-r}(0,\zeta)|_{L^2(\sigma,\eta)}+\frac{C}{\sqrt{\gamma}}\sum_{t\in\mathbb{Z}\setminus 0}|\alpha_r\alpha_t D(\epsilon,k,k-r;\beta_l)V_{k-r-t}|_{L^2}\right].
\end{aligned}
\tag{3.70}
$$

The estimate (3.70) shows that the piece of the term on the right of (3.51)(b) given by $\frac{C}{\gamma}\left|\frac{\chi_b^2(\epsilon,k,k-r)\alpha_r V_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2}$ can be *replaced* by (not dominated by)

$$\frac{C}{\gamma}\sum_{t\in\mathbb{Z}}|\alpha_r\alpha_t D(\epsilon,k,k-r;\beta_l)\mathcal{W}_{k-r-t}|_{L^2}\,. \tag{3.71}$$

The piece of the term on the right of (3.51)(a) given by $\frac{C}{\gamma}\left|\frac{\chi_b^2(\epsilon,k,k-r)\alpha_r V_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2}$ arises from the following part of "$\frac{\chi_b^2(\epsilon,k,k-r)}{\Delta(\epsilon,k)}$(3.41)":

$$\frac{\chi_b^2(\epsilon,k,k-r)}{\Delta(\epsilon,k)}\int_{x_2}^{\infty}e^{i\xi_+(\epsilon,k)(x_2-s)+i\frac{r\omega_N(\beta_l)}{\epsilon}s}\alpha_r M^+(\epsilon,k,k-r)V_{k-r}(s,\zeta)ds, \tag{3.72}$$

The replacement, determined by essentially the same argument as given above, is again (3.71).

7. Improving the $\left|\frac{\chi_b^1(\epsilon,k,k-r)\alpha_r w^-_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2}$ **pieces of** (3.51)**.** First we take $\beta=\beta_l$. For a given r the main contribution to the second term (in parentheses) on the right in (3.51)(b) arises from the part of (3.39) given by

$$A:=e^{i\xi_-(\epsilon,k)x_2}[Br_-(\epsilon,k)]^{-1}Br_+(\epsilon,k)\int_0^{\infty}e^{i\xi_+(\epsilon,k)(-s)+i\frac{r\omega_N(\beta_l)}{\epsilon}s}\alpha_r b(\epsilon,k,k-r)w^-_{k-r}(s,\zeta)]ds. \tag{3.73}$$

Ignoring some factors in the integrand that are $O(1)$ and independent of s, we see that the $p-$component of A is a sum of terms of the form

$$A_{p,i,j}:=e^{i\omega_p(\epsilon,k)x_2}\frac{1}{\Delta(\epsilon,k)}\int_0^{\infty}e^{i\omega_i(\epsilon,k)(-s)+i\frac{r\omega_N(\beta_l)}{\epsilon}s}\alpha_r w^-_{k-r,j}(s,\zeta)ds \tag{3.74}$$

where $p\in\mathcal{I}$, $i\in\mathcal{O}$, and $w^-_{k-r,j}$, $j\in\mathcal{I}$ denotes a component of w^-_{k-r}.

For any $(i,j)\in\mathcal{O}\times\mathcal{I}$ recall the condition (3.16) involving the function $E_{i,j}(\epsilon,k,k-r;\beta_l):=\omega_i(\epsilon,k)-r\omega_N(\beta_l/\epsilon)-\omega_j(\epsilon,k-r)$:

$$|E_{i,j}(\epsilon,k,k-r)|\geq C_3\frac{|X_k|}{|r|}\text{ or }|E_{i,j}(\epsilon,k,k-r)|\geq C_3|X_{k-r}|\text{ on supp }\chi_b^1(\epsilon,k,k-r). \tag{3.75}$$

To estimate $\chi_b^1(\epsilon,k,k-r;\beta_l)A_{p,i,j}$ we first decompose[46]

$$\chi_b^1(\epsilon,k,k-r)=\chi_G^1(\epsilon,k,k-r)+\chi_B^1(\epsilon,k,k-r) \tag{3.76}$$

[46]The cutoffs on the right in (3.76) have additional dependence on (i,j), which we suppress in the notation.

into "good" and "bad" pieces supported respectively where the condition (3.75) holds, does not hold.

We proceed to improve the estimate of $\chi^1_G(\epsilon,k,k-r)A_{p,i,j}$. We decompose $\chi^1_G(\epsilon,k,k-r)$,

$$\chi^1_G(\epsilon,k,k-r)=\chi^1_{G,I}(\epsilon,k,k-r)+\chi^1_{G,II}(\epsilon,k,k-r)+\chi^1_{G,III}(\epsilon,k,k-r), \tag{3.77}$$

into pieces where respectively, (I) the first alternative in (3.75) holds, (II) the first alternative fails and $|X_{k-r}|\geq\frac{|X_k|}{N_2}$, (III) the first alternative fails and $|X_{k-r}|<\frac{|X_k|}{N_2}$; $N_2\in\mathbb{N}$ is chosen large enough below.

In case (I) we have

$$\left|\frac{1}{\Delta(\epsilon,k)E_{i,j}}\right|\leq\frac{|X_k|}{\gamma}\frac{|r|}{C_3|X_k|}\sim\frac{|r|}{\gamma}, \tag{3.78}$$

so an integration by parts just like that in step **6** gives[47]

$$\begin{aligned}&|\chi^1_{G,I}(\epsilon,k,k-r)A_{p,i,j}|_{L^2}\lesssim\\&\frac{C}{\gamma^{3/2}}\left[|\alpha_r D(\epsilon,k,k-r;\beta_l)w^-_{k-r,j}(0,\zeta)|_{L^2(\sigma,\eta)}+\frac{C}{\sqrt{\gamma}}\sum_{t\in\mathbb{Z}\setminus 0}|\alpha_r\alpha_t D(\epsilon,k,k-r;\beta_l)w_{k-r-t}|_{L^2}\right].\end{aligned} \tag{3.79}$$

In case (II) we find

$$\left|\frac{1}{\Delta(\epsilon,k)E_{i,j}}\right|\leq\frac{|X_k|}{\gamma}\frac{1}{C_3|X_{k-r}|}\sim\frac{N_2}{\gamma}, \tag{3.80}$$

so a similar integration by parts yields an estimate just like (3.79) for $|\chi^1_{G,II}(\epsilon,k,k-r)A_{p,i,j}|_{L^2}$.

In case (III) the first two terms in the expression for $E_{i,j}$ are dominant, so we can use the argument of step 6 (recall (3.62)) to show

$$|E^{-1}_{i,j}|\leq\frac{C(N_2)}{|X_k|},\text{ and hence }\left|\frac{1}{\Delta(\epsilon,k)E_{i,j}}\right|\leq\frac{C(N_2)}{\gamma}, \tag{3.81}$$

provided N_2 is large enough. Thus, we get the estimate (3.79) for $|\chi^1_{G,III}(\epsilon,k,k-r)A_{p,i,j}|_{L^2}$,

To estimate $\chi^1_B(\epsilon,k,k-r)A_{p,i,j}$ we do a direct estimate using (3.74), and the fact that[48]

$$|\Delta(\epsilon,k|^{-1}\lesssim\frac{|X_k|}{\gamma}\leq\frac{C_4|r|}{\epsilon\gamma}\text{ on supp }\chi^1_B(\epsilon,k,k-r),\text{ for }C_4\text{ as in (3.17).} \tag{3.82}$$

Thus, we obtain

$$|\chi^1_B(\epsilon,k,k-r)A_{p,i,j}|_{L^2}\leq\frac{C}{\gamma}\frac{C_4|r|}{\epsilon\gamma}|w^-_{k-r}|_{L^2}. \tag{3.83}$$

These estimates show that the piece of the second term on the right of (3.51)(b) given by $\frac{C}{\gamma}\left|\frac{\chi^1_b(\epsilon,k,k-r)\alpha_r w^-_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2}$ can be *replaced* by (not dominated by) (3.71).

[47] In this integration by parts $w^{*,-}_{k-r,j}(x_2,\zeta):=e^{-i\omega_j(\epsilon,k-r)x_2}w^-_{k-r,j}$ replaces V^*_{k-r} as in (3.65), and $h^-_{k-r,j}$, the j-component of the right side of $(3.36)_{k-r}$, replaces h_{k-r} as in (3.67).

[48] Use Lemma 3.6 here.

The piece of the second term on the right of (3.51)(a) given by $\frac{C}{\gamma}\left|\frac{\chi_b^1(\epsilon,k,k-r)\alpha_r w^-_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2}$ arises from the following part of "$\frac{\chi_b^1(\epsilon,k,k-r)}{\Delta(\epsilon,k)}$(3.38)":

$$\frac{\chi_b^1(\epsilon,k,k-r)}{\Delta(\epsilon,k)}\int_{x_2}^{\infty} e^{i\xi_+(\epsilon,k)(x_2-s)+i\frac{\omega_N(\beta_l)}{\epsilon}s}\alpha_r b(\epsilon,k,k-r)w^-_{k-r}(s,\zeta)ds, \tag{3.84}$$

The replacement, determined by essentially the same argument as given above, is again (3.71).

Remark 3.19. *Proposition 3.22 below implies that $\chi^1_B(\epsilon,k,k-r;\beta_l)$ can be nonzero* only *in the cases $i\in\mathcal{O}$, $j\in\mathcal{I}\setminus\{N\}$. Equivalently, the function $E_{i,N}$ satisfies condition* (3.75) *for all $i\in\mathcal{O}$. This observation is used in the proof of Proposition 4.3.*

We now conclude step **7** by treating the case $\beta\neq\beta_l$. Proposition 3.10 shows that on the support of $\chi_b^1(\epsilon,k,k-r;\beta)$ we have

$$|E_{i,j}(\epsilon,k,k-r;\beta)|\geq C_3|X_{k-r}|. \tag{3.85}$$

Thus, this case can be treated just like the term involving $\chi^1_{G,II}(\epsilon,k,k-r)$ above, except that now we use $|X_{k-r}|\gtrsim\frac{|X_k|}{|r|}$ in place of $|X_{k-r}|\gtrsim\frac{|X_k|}{N_2}$. Again we obtain an estimate like (3.79), but with factors $D(\epsilon,k,k-r;\beta)$ appearing on the right.

8. Improving the $\left|\frac{\chi_b^1(\epsilon,k,k-r)\alpha_r w^+_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2}$ pieces of (3.51)**.** Again we begin with the case $\beta=\beta_l$. To treat the pieces involving w^+_{k-r} on the right in (3.51), we first decompose $\chi_b^1(\epsilon,k,k-r;\beta_l)$,

$$\chi_b^1(\epsilon,k,k-r)=\chi^1_{b,\mathbb{I}}(\epsilon,k,k-r)+\chi^1_{b,\mathbb{III}}(\epsilon,k,k-r), \tag{3.86}$$

into pieces where respectively, ($\mathbb{I}$) $|X_{k-r}|\geq\frac{|X_k|}{N_3}$, ($\mathbb{III}$) $|X_{k-r}|<\frac{|X_k|}{N_3}$, for $N_3\in\mathbb{N}$ to be chosen large below. In case ($\mathbb{I}$) we insert $\frac{\Delta(\epsilon,k-r)}{\Delta(\epsilon,k-r)}$ and use (recall (3.13)(b) and Remark 3.7)

$$\left|\frac{\Delta(\epsilon,k-r)}{\Delta(\epsilon,k)}\right|\approx\frac{|X_k|}{|X_{k-r}|}\leq N_3 \text{ on supp }\chi_b^1(\epsilon,k,k-r) \tag{3.87}$$

to obtain

$$\frac{C}{\gamma}\left|\frac{\chi^1_{b,\mathbb{I}}(\epsilon,k)\alpha_r w^+_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2}\leq\frac{C}{\gamma}|\alpha_r D(\epsilon,k,k-r;\beta_l)\tilde{w}^+_{k-r}|_{L^2}. \tag{3.88}$$

In case ($\mathbb{III}$) consider the contribution to the term involving w^+_{k-r} on the right in (3.51)(a) given by

$$\frac{\chi^1_{b,\mathbb{III}}(\epsilon,k,k-r)}{\Delta(\epsilon,k)}\int_{x_2}^{\infty} e^{i\xi_+(\epsilon,k)(x_2-s)+i\frac{r\omega_N(\beta_l)}{\epsilon}s}\alpha_r a(\epsilon,k,k-r)w^+_{k-r}(s,\zeta)ds. \tag{3.89}$$

If we define the $|\mathcal{O}|\times|\mathcal{O}|$ matrix

$$F_{+,+}(\epsilon,k,k-r)=\xi_+(\epsilon,k)-r\omega_N(\beta_l/\epsilon)I_{|\mathcal{O}|}-\xi_+(\epsilon,k-r), \tag{3.90}$$

we see by an argument parallel to (3.59)-(3.62) that the first two terms on the right of (3.90) are dominant for large N_1, so

$$\left|\frac{1}{\Delta(\epsilon,k)}F^{-1}_{+,+}(\epsilon,k,k-r)\right| \lesssim \frac{1}{\gamma} \text{ on supp } \chi^1_{b,\mathbb{III}}(\epsilon,k,k-r). \tag{3.91}$$

Thus, an integration by parts similar to that in step **6** shows that the piece of the term on the right of (3.51)(a) given by $\frac{C}{\gamma}\left|\frac{\chi^1_{b,\mathbb{III}}(\epsilon,k,k-r)\alpha_r w^+_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2}$ can be *replaced* by (not dominated by)

$$\frac{C}{\gamma}\sum_{t\in\mathbb{Z}}|\alpha_r\alpha_t D(\epsilon,k,k-r;\beta_l)\mathcal{W}_{k-r-t}|_{L^2}\,. \tag{3.92}$$

Finally, we must consider the contribution in case ($\mathbb{III}$) to the term involving w^+_{k-r} on the right in (3.51)(b) given by

(3.93)
$$e^{i\xi_-(\epsilon,k)x_2}\chi^1_{b,\mathbb{III}}(\epsilon,k,k-r)[Br_-(\epsilon,k)]^{-1}Br_+(\epsilon,k)\int_0^\infty e^{i\xi_+(\epsilon,k)(-s)+ir\frac{\omega_N(\beta_l)}{\epsilon}s}\alpha_r a(\epsilon,k,k-r)w^+_{k-r}(s,\zeta)ds.$$

The replacement, determined by essentially the same argument involving $F_{+,+}$, is again (3.92).

We conclude step **8** by treating the case $\beta\neq\beta_l$. We again insert $\frac{\Delta(\epsilon,k-r)}{\Delta(\epsilon,k-r)}$, but now we use Remark 3.14 and Definition 3.15 to obtain parallel to (3.88)

$$\frac{C}{\gamma}\left|\frac{\chi^1_b(\epsilon,k)\alpha_r w^+_{k-r}}{\Delta(\epsilon,k)}\right|_{L^2} \leq \frac{C}{\gamma}|\alpha_r D(\epsilon,k,k-r;\beta)\tilde{w}^+_{k-r}|_{L^2}. \tag{3.94}$$

9. Using Proposition 3.5 and the formulas (3.38), (3.39), it is easy to see that $\frac{1}{\sqrt{\gamma}}|\tilde{w}^+_k(0,\zeta;\beta)|_{L^2(\sigma,\eta)}$ and $\frac{1}{\sqrt{\gamma}}|w^-_k(0,\zeta;\beta))|_{L^2(\sigma,\eta)}$ satisfy, respectively, the same estimates as $|\tilde{w}^+_k(x_2,\zeta;\beta)|_{L^2}$, $|w^-_k(x_2,\zeta,\beta)|_{L^2}$. Combining the estimates of steps **4-8** establishes for all $\beta\in\Upsilon^+_0$ the estimate (3.50) for the case $F=0$.

10. If $F(t,x,\theta)=\sum_{k\in\mathbb{Z}}F_k(t,x)e^{ik\theta}$, the only modification to (3.38), (3.39) is to add

$$-i\int_{x_2}^\infty e^{i\xi_+(\epsilon,k)(x_2-s)}\ell_+(\epsilon,k)\widehat{F_k}(s,\tau,\eta)ds, \tag{3.95}$$

to the right side of (3.38), and to add the terms

$$\begin{gathered}i\int_0^{x_2} e^{i\xi_-(\epsilon,k)(x_2-s)}\ell_-(\epsilon,k)\widehat{F_k}(s,\tau,\eta)ds+\\ ie^{i\xi_-(\epsilon,k)x_2}[Br_-(\epsilon,k)]^{-1}Br_+(\epsilon,k)\int_0^\infty e^{i\xi_+(\epsilon,k)(-s)}\ell_+(\epsilon,k)\widehat{F_k}(s,\tau,\eta)ds\end{gathered} \tag{3.96}$$

to the right side of (3.39). Using step **9** and applying Proposition 3.5 again, we obtain the estimate (3.48).

□

Remark 3.20. *In the case where the oscillatory coefficient in* (3.31) *has only positive spectrum, if one takes* $F(t,x,\theta) = \sum_{k=1}^{\infty} F_k(t,x)e^{ik\theta}$ *and* $G(t,x_1,\theta) = \sum_{k=1}^{\infty} G_k(t,x_1)e^{ik\theta}$*, then the solution to* (3.31) *satisfies* $V_k = 0$ *for* $k < 1$*. The iteration estimate* (3.48) *then reduces to*

$$\|V_k\| \leq \frac{C}{\gamma}\sum_{r=1}^{k-1}\sum_{t=0}^{k-r-1} \|\alpha_r\alpha_t\mathbb{D}(\epsilon,k,k-r)V_{k-r-t}\| + \frac{C}{\gamma^2}\left|\widehat{F_k}|X_k|\right|_{L^2} + \frac{C}{\gamma^{3/2}}\left|\widehat{G_k}|X_k|\right|_{L^2(\sigma,\eta)}. \tag{3.97}$$

The obvious analogue of the estimate (3.97) *holds in the case where* $F(t,x,\theta) = \sum_{k=N^*}^{\infty} F_k(t,x)e^{ik\theta}$ *and* $G(t,x_1,\theta) = \sum_{k=N^*}^{\infty} G_k(t,x_1)e^{ik\theta}$ *for any* $N^* \in \mathbb{Z}$*. Moreover, the proof of Proposition 3.18 shows that the constants* C*,* γ_0 *appearing there are independent of* N^**. This remark is used in the cascade estimates of section 4.*

3.5 Control of amplification factors

In preparation for the proof of Proposition 3.10, we recall that if the sequence $(k_p)_{p\in\mathbb{Z}}$ is admissible (Definition 3.9), for $(\zeta,\epsilon) \in \Xi \times (0,\epsilon_0]$ and $\beta \in \Upsilon_0^+$ we define the function of ζ:

$$E_{i,j}(\epsilon,k_p,k_{p-1};\beta) := \omega_i(\epsilon,k_p;\beta) - \frac{r_p\omega_N(\beta_l)}{\epsilon} - \omega_j(\epsilon,k_{p-1};\beta), \text{ where } i \in \mathcal{O},\ j \in \mathcal{I} \tag{3.98}$$

Here, we have $\omega_i(\epsilon,k;\beta)(\zeta) = \omega_i(\zeta;\beta)|_{\zeta=X_k}$, and, moreover, $\omega_N(\beta_l) = \omega_N(\zeta;\beta_l)|_{\zeta=\beta_l}$.

Proof of Proposition 3.10. We will carry out the proof for a strictly increasing sequence (k_p), so $r_p > 0$ for all p; the decreasing case is similar. Recall that $X_k = \zeta + \frac{k\beta_l}{\epsilon}$.

For $\beta, \beta_l \in \Upsilon_0^+$ let $\alpha = \alpha(\beta) \in [0, \frac{\pi}{2}]$ be the angle between β and β_l. When a fixed β as in (3.98) is being considered, we define the positive "vertical" direction to be the β direction, and the positive "horizontal" direction to be the direction of a vector obtained by rotating β by $\frac{\pi}{2}$ clockwise. For each $X_k \in \Xi$ there is a point $\tilde{X}_k := (X_k \cdot \beta)\beta$ closest to it on the line $\mathcal{L}(\beta) := \{t\beta : t \in \mathbb{R}\}$.[49] We have

$$X_{k_j} = X_{k_{j-1}} + \frac{r_j\beta_l}{\epsilon}, \quad \tilde{X}_{k_j} = \tilde{X}_{k_{j-1}} + \frac{(r_j\cos\alpha)\beta}{\epsilon} \text{ for all } j. \tag{3.99}$$

For example if $\beta \perp \beta_l$, we have $\tilde{X}_{k_j} = \tilde{X}_{k_{j-1}}$. In general the vertical displacement in passing from X_{k_j} to $X_{k_{j-1}}$ is $-\frac{r_j\cos\alpha}{\epsilon}$, while the horizontal displacement is $\pm\frac{r_j\sin\alpha}{\epsilon}$ (recall $|\beta_l| = 1$).[50]

Moreover, for $\delta > 0$ small we have

$$|X_{k_j} - \tilde{X}_{k_j}| \leq 2\frac{\delta}{r_j}|\tilde{X}_{k_j}| \Rightarrow (1-2\frac{\delta}{r_j})|\tilde{X}_{k_j}| \leq |X_{k_j}| \leq (1+2\frac{\delta}{r_j})|\tilde{X}_{k_j}| \text{ for } X_{k_j} \in \Gamma_{\frac{\delta}{r_j}}(\beta). \tag{3.100}$$

1. Let $\alpha_1 > 0$ be the smallest of the angles $\alpha(\beta)$ for $\beta \in \Upsilon_0^+ \setminus \{\beta_l\}$. Fix such a β and let $\tilde{\Gamma}(p)$ be the portion of $\Gamma_{\frac{\delta}{r_p}}(\beta)$ bounded "above" by the plane orthogonal to β containing X_{k_p} and "below" by the plane orthogonal to β containing $X_{k_{p-1}}$. Also, let $\tilde{X}_{k_{p-1}} = t(p)\frac{\beta}{\epsilon}$ be the orthogonal projection of $X_{k_{p-1}}$ on β. Then the maximum width of $\tilde{\Gamma}(p)$ is $\lesssim 2\frac{\delta}{r_p}(|t(p)| + r_p)/\epsilon$, while the "horizontal" displacement in passing from X_{k_p} to $X_{k_{p-1}}$ is $\frac{r_p\sin\alpha}{\epsilon} \geq \frac{r_p\sin\alpha_1}{\epsilon}$. Suppose

$$|t(p)| \lesssim r_p. \tag{3.101}$$

[49] Here interpret $\zeta\cdot\beta$ as $(\sigma,\gamma,\eta)\cdot(\beta_0,0,\beta_1)$, where $\beta = (\beta_0,\beta_1)$.

[50] More precisely, this is the horizontal displacement of the projection of X_{k_j} into the (σ,η) plane.

Then X_{k_p} and $X_{k_{p-1}}$ cannot both lie in $\Gamma_{\frac{\delta}{r_p}}(\beta)$ if $\delta = \delta(\alpha_1)$ is small enough. The case where (3.101) does not hold is treated at the end of step **5**.

2. For any fixed $\beta \in \Upsilon_0^+$ we assume now that X_{k_p} and $X_{k_{p-1}}$ lie in $\Gamma_{\frac{\delta}{r_p}}(\beta)$. With $\omega_i(\zeta) = \omega_i(\zeta;\beta)$, write $E_{i,j}$ as in (3.98) as

$$
\begin{aligned}
&E_{i,j}(\epsilon, k_p, k_{p-1}) = \omega_i(X_{k_p}) - \frac{r_p\omega_N(\beta_l)}{\epsilon} - \omega_j(X_{k_{p-1}}) \\
&\tilde{E}_{i,j}(\epsilon, k_p, k_{p-1}) := \omega_i(\tilde{X}_{k_p}) - \frac{r_p\omega_N(\beta_l)}{\epsilon} - \omega_j(\tilde{X}_{k_{p-1}})
\end{aligned} \tag{3.102}
$$

for $\tilde{X}_{k_p}$ as in (3.99). We also have

$$
\begin{aligned}
&\tilde{X}_{k_{p-1}} = t\frac{\beta}{\epsilon} \text{ and } \tilde{X}_{k_p} = \tilde{X}_{k_{p-1}} + \frac{(r_p\cos\alpha)\beta}{\epsilon} = (t + r_p\cos\alpha)\frac{\beta}{\epsilon} \text{ for some } t = t(p) \in \mathbb{R}, \\
&X_{k_p} = X_{k_{p-1}} + \frac{r_p\beta_l}{\epsilon}.
\end{aligned} \tag{3.103}
$$

3. Since $\omega_j(\zeta)$ is positively homogeneous of degree one, it follows from the definition of $\omega_j(\zeta)$ on $\Gamma_\delta(\beta)$ that in fact

$$\omega_j(s\beta) = s\omega_j(\beta) \text{ for all } s \in \mathbb{R}.$$

We compute

$$
\begin{aligned}
&\tilde{E}_{i,j}(\epsilon, k_p, k_{p-1}) = \omega_i\left((t + r_p\cos\alpha)\frac{\beta}{\epsilon}\right) - \frac{r_p\omega_N(\beta_l)}{\epsilon} - \omega_j\left(t\frac{\beta}{\epsilon}\right) = \\
&\frac{t}{\epsilon}(\omega_i(\beta) - \omega_j(\beta)) + \frac{(r_p\cos\alpha)\omega_i(\beta) - r_p\omega_N(\beta_l)}{\epsilon}.
\end{aligned} \tag{3.104}
$$

Since $\omega_i(\beta) - \omega_j(\beta) \neq 0$, we see that there exists a $t_p(\beta) \in \mathbb{R}$ such that the right side vanishes at $t = t_p(\beta)$. Namely,

$$t_p(\beta) = r_p\,\Omega_{i,j}(\beta), \text{ where } \Omega_{i,j}(\beta) := \frac{\cos\alpha\;\omega_i(\beta) - \omega_N(\beta)}{\omega_j(\beta) - \omega_i(\beta)}. \tag{3.105}$$

Writing $t = (t - t_p) + t_p$ we obtain

$$\tilde{E}_{i,j}(\epsilon, k_p, k_{p-1}) = \frac{(t - t_p)}{\epsilon}(\omega_i(\beta) - \omega_j(\beta)) := \frac{(t - t_p)}{\epsilon}C(\beta). \tag{3.106}$$

4. Fix $\lambda > 0$ and suppose $|t - t_p| \geq \lambda$. Then (3.106) implies

$$|\tilde{E}_{i,j}(\epsilon, k_p, k_{p-1})| \geq \lambda\frac{C(\beta)}{\epsilon}. \tag{3.107}$$

If the condition $|t| \geq 2|t_p| + 2$ holds, then we have

$$|\tilde{E}_{i,j}(\epsilon, k_p, k_{p-1})| \geq \frac{|t|}{2}\frac{C(\beta)}{\epsilon}. \tag{3.108}$$

5. Using (3.102) and (3.100) we obtain

$$\begin{aligned}|E_{i,j}(\epsilon,k_p,k_{p-1})-\tilde{E}_{i,j}(\epsilon,k_p,k_{p-1})| &\leq |\omega_i(X_{k_p})-\omega_i(\tilde{X}_{k_p})|+|\omega_j(X_{k_{p-1}})-\omega_j(\tilde{X}_{k_{p-1}})| \\ &\lesssim |X_{k_p}-\tilde{X}_{k_p}|+|X_{k_{p-1}}-\tilde{X}_{k_{p-1}}|,\end{aligned} \tag{3.109}$$

and thus

$$|E_{i,j}(\epsilon,k_p,k_{p-1})| \geq |\tilde{E}_{i,j}(\epsilon,k_p,k_{p-1})| - C(|X_{k_p}-\tilde{X}_{k_p}|+|X_{k_{p-1}}-\tilde{X}_{k_{p-1}}|). \tag{3.110}$$

Since $X_{k_p}, X_{k_{p-1}} \in \Gamma_{\frac{\delta}{r_p}}(\beta)$ we have

$$|X_{k_p}-\tilde{X}_{k_p}| \lesssim \frac{\delta}{r_p}|\tilde{X}_{k_p}| \text{ and } |X_{k_{p-1}}-\tilde{X}_{k_{p-1}}| \lesssim \frac{\delta}{r_p}|\tilde{X}_{k_{p-1}}|. \tag{3.111}$$

If $|\tilde{X}_{k_{p-1}}| \leq \frac{2|t_p|+2}{\epsilon}$, we deduce from (3.107), (3.110),(3.111) that for $\delta=\delta(\lambda)$ small and $|t-t_p|\geq\lambda$:

$$|E_{i,j}(\epsilon,k_p,k_{p-1})| \geq \frac{\lambda}{2}\frac{C(\beta)}{\epsilon} \geq C_3(\lambda)\frac{|X_{k_p}|}{r_p}. \tag{3.112}$$

If $|\tilde{X}_{k_{p-1}}| \geq \frac{2|t_p|+2}{\epsilon}$, then $|t|\geq 2|t_p|+2$, and we deduce from (3.108), (3.110), (3.111) that for δ small:

$$|E_{i,j}(\epsilon,k_p,k_{p-1})| \geq \frac{|t|C(\beta)}{3\epsilon} \geq C_3(\lambda)|X_{k_{p-1}}|, \tag{3.113}$$

after reducing $C_3(\lambda)$ if necessary.

Step **1** implies that if $\beta\neq\beta_l$ and δ_0 is small enough, we *must* have $|\tilde{X}_{k_{p-1}}| \geq \frac{2|t_p|+2}{\epsilon}$; otherwise, we could not have both X_{k_p} and $X_{k_{p-1}}$ in $\Gamma_{\frac{\delta}{r_p}}(\beta)$. From (3.99) we see that

$$\frac{|\tilde{X}_{k_p}|}{|\tilde{X}_{k_{p-1}}|} \leq 1+O(r_p), \tag{3.114}$$

so (3.100) implies $|X_{k_{p-1}}| \gtrsim \frac{|X_{k_p}|}{|r_p|}$. With (3.113) this concludes the proof of part 1 of the proposition.

6. Assume now that $\beta=\beta_l$. For any given (ζ,ϵ) and δ, C_3 as in step **5**, we define $M_{i,j}(\zeta,\epsilon,\delta;C_3)$ by

$$M_{i,j}(\zeta,\epsilon,\delta;C_3) := \{p\in\mathbb{Z} : X_{k_p}\in\Gamma_{\frac{\delta}{r_p}}(\beta_l), X_{k_{p-1}}\in\Gamma_{\frac{\delta}{r_p}}(\beta_l), \text{ and both (3.112), (3.113) fail.}\} \tag{3.115}$$

The above estimates show that

$$M_{i,j}(\zeta,\epsilon,\delta;C_3) \subset \mathcal{M}_{i,j}(\zeta,\epsilon,\delta) := \{p\in\mathbb{Z} : X_{k_p}\in\Gamma_{\frac{\delta}{r_p}}(\beta_l), X_{k_{p-1}}\in\Gamma_{\frac{\delta}{r_p}}(\beta_l), |t-t_p|<\lambda\}. \tag{3.116}$$

Since $|t_p(\beta_l)|\lesssim r_p$, the estimate (3.17) now follows directly from the definition of $\mathcal{M}_{i,j}$ and (3.103). $\square$

Proof of Proposition 3.11. We carry out the proof for a strictly increasing sequence (k_j); the decreasing case is similar. We fix $\beta \in \Upsilon_0^+ \setminus \{\beta_l\}$ and continue to use the notation introduced just before step **1** in the previous proof. By symmetry of $\Gamma_\delta := \Gamma_\delta(\beta)$ about the line $\mathcal{L}(\beta) := \{t\beta : t \in \mathbb{R}\}$, it will suffice to consider the case where the angle $\alpha = \alpha(\beta) \in (0, \frac{\pi}{2}]$ from β_l to β is counterclockwise.

1. If $X_{k_j} \notin \Gamma_{\frac{\delta}{2r_j}}(\beta)$, then by Lemma 3.6 (a),(g) we have

$$\left|\frac{\Delta(\epsilon, k_{j-1})}{\Delta(\epsilon, k_j)}\right| \lesssim r_j. \tag{3.117}$$

2. Now suppose

$$X_{k_j} \in \Gamma_{\frac{\delta}{2r_j}}(\beta), \;\; X_{k_{j-1}} \notin \Gamma_{\frac{\delta}{r_j}}(\beta). \tag{3.118}$$

If $|X_{k_j}| \leq \frac{2r_j}{\epsilon}$, then from (3.26) and Lemma 3.6(a) we obtain

$$\left|\frac{\Delta(\epsilon, k_{j-1})}{\Delta(\epsilon, k_j)}\right| \lesssim \frac{r_j}{\epsilon\gamma}. \tag{3.119}$$

Next assume $|X_{k_j}| \geq \frac{2r_j}{\epsilon}$ and take $X_{k_j} \in \Gamma^+_{\frac{\delta}{2r_j}}(\beta)$.[51] For δ small the width of $\Gamma^+_{\frac{\delta}{2r_j}}(\beta)$ at the "height" $|\tilde{X}_{k_j}|$ is $\approx |X_{k_j}|\frac{2\delta}{2r_j} = |X_{k_j}|\frac{\delta}{r_j}$. The horizontal and vertical displacements in passing from X_{k_j} to $X_{k_{j-1}}$ are respectively $\frac{-r_j \sin\alpha}{\epsilon}$, $\frac{-r_j\cos\alpha}{\epsilon}$, so the width of the larger cone $\Gamma^+_{\frac{\delta}{r_j}}(\beta)$ at the height $|\tilde{X}_{k_{j-1}}|$ is $\approx (|X_{k_j}| - r_j\cos\alpha)\frac{2\delta}{r_j}$. Since $X_{k_{j-1}} \notin \Gamma_{\frac{\delta}{r_j}}(\beta)$, we have the following lower bound for the magnitude of the horizontal displacement in passing from X_{k_j} to $X_{k_{j-1}}$:

$$\frac{r_j \sin\alpha}{\epsilon} \gtrsim (|X_{k_j}| - r_j\cos\alpha)\frac{\delta}{r_j} - |X_{k_j}|\frac{\delta}{2r_j}. \tag{3.120}$$

Hence

$$|X_k| \lesssim \frac{r_j^2}{\epsilon} \Rightarrow \left|\frac{\Delta(\epsilon, k_{j-1})}{\Delta(\epsilon, k_j)}\right| \lesssim \frac{r_j^2}{\epsilon\gamma}. \tag{3.121}$$

Observe also that if $\delta < \alpha$, for a fixed (ζ, ϵ) the condition (3.118) can be satisfied for just *one* k_j in the admissible sequence.

3. Suppose finally that

$$X_{k_j} \in \Gamma_{\frac{\delta}{2r_j}}(\beta), \;\; X_{k_{j-1}} \in \Gamma_{\frac{\delta}{r_j}}(\beta). \tag{3.122}$$

Writing $X_{k_j} = X_k$, $X_{k_{j-1}} = X_{k-r}$ now, by Lemma (3.6)(b) we have

$$\begin{aligned} \left|\frac{\Delta(\epsilon, k-r)}{\Delta(\epsilon, k)}\right| &\approx \frac{|X_k|}{|X_{k-r}|}\frac{|\tau + (k-r)\frac{\sigma_l}{\epsilon} - c_+(\beta)(\eta + (k-r)\frac{\eta_l}{\epsilon})|}{|\tau + k\frac{\sigma_l}{\epsilon} - c_+(\beta)(\eta + k\frac{\eta_l}{\epsilon})|} \leq \\ &\frac{|X_k|}{|X_{k-r}|}\left(1 + \frac{r|\frac{(\sigma_l - c_+(\beta)\eta_l)}{\epsilon}|}{|\tau + k\frac{\sigma_l}{\epsilon} - c_+(\beta)(\eta + k\frac{\eta_l}{\epsilon})|}\right). \end{aligned} \tag{3.123}$$

[51]The case $X_{k_j} \in \Gamma^-_{\frac{\delta}{2r_j}}(\beta)$ is treated similarly.

In the unlikely case that $c_+(\beta)=c_+(\beta_l)$, the right side of (3.123) is $\frac{|X_k|}{|X_{k-r}|}$, and this is $\lesssim r$ by part 1 of Proposition 3.10. Now suppose $c_+(\beta)\neq c_+(\beta_l)$ and set $m(\beta):=c_+(\beta)\eta_l-\sigma_l\neq 0$. If

$$|\tau+k\frac{\sigma_l}{\epsilon}-c_+(\beta)(\eta+k\frac{\eta_l}{\epsilon})|=\left|(\tau-c_+(\beta)\eta)-k\frac{m(\beta)}{\epsilon}\right|\geq\frac{|m(\beta)|}{2\epsilon}, \tag{3.124}$$

then (3.123) implies $\left|\frac{\Delta(\epsilon,k-r)}{\Delta(\epsilon,k)}\right|\lesssim r^2$. The other possibility is that

$$\gamma\leq\left|(\tau-c_+(\beta)\eta)-k\frac{m(\beta)}{\epsilon}\right|<\frac{|m(\beta)|}{2\epsilon}, \tag{3.125}$$

and in this case (3.123) implies $\left|\frac{\Delta(\epsilon,k-r)}{\Delta(\epsilon,k)}\right|\lesssim\frac{r^2}{\epsilon\gamma}$. Observe also that for a fixed choice of ζ, the condition (3.125) can hold for at most one choice of $k\in\mathbb{Z}$.

4. The above three steps imply that for fixed (ζ,ϵ) we have $\left|\frac{\Delta(\epsilon,k_{j-1})}{\Delta(\epsilon,k_j)}\right|\lesssim r_j^2$ except for at most two choices of j, and in the exceptional cases we have $\left|\frac{\Delta(\epsilon,k_{j-1})}{\Delta(\epsilon,k_j)}\right|\lesssim\frac{r_j^2}{\epsilon\gamma}$. [52]

□

The next Proposition is used in the proof of part (a) of Theorem 2.11.

Proposition 3.21. *Suppose $i\in\mathcal{O}$, $j\in\mathcal{I}\setminus\{N\}$ and assume that*

$$\Omega_{i,j}(\beta_l):=\frac{\omega_i(\beta_l)-\omega_N(\beta_l)}{\omega_j(\beta_l)-\omega_i(\beta_l)}\in(-1,0). \tag{3.126}$$

There exist positive constants ϵ_0, δ_0 and positive constants C_3, C_4 independent of $(\zeta,\epsilon,p)\in\Xi\times(0,\epsilon_0]\times\mathbb{Z}$ *such that the following situation holds:*

Let (k_p) be an admissible sequence. For any given $(\zeta,\epsilon,\delta)\in\Xi\times(0,\epsilon_0]\times(0,\delta_0]$ there exists at most one exceptional element $m=m^{i,j}(\zeta,\epsilon,\delta)\in\mathbb{Z}$, such that if $p\neq m$ and if $X_{k_p}\in\Gamma_{\frac{\delta}{|r_p|}}(\beta_l)$, $X_{k_{p-1}}\in\Gamma_{\frac{\delta}{|r_p|}}(\beta_l)$, then we have either

$$|E_{i,j}(\epsilon,k_p,k_{p-1};\beta_l)|\geq C_3\frac{|X_{k_p}|}{|r_p|}\ \text{or}\ |E_{i,j}(\epsilon,k_p,k_{p-1};\beta_l)|\geq C_3|X_{k_{p-1}}|. \tag{3.127}$$

Moreover, the exceptional value m satisfies

$$|X_{k_m}|\leq\frac{C_4|r_m|}{\epsilon}. \tag{3.128}$$

Proof. We carry out the proof for a strictly increasing sequence (k_p); the decreasing case is similar.

1. Consider $\mathcal{M}_{i,j}(\zeta,\epsilon,\delta)$ as in step **6** of the previous proof and $\delta(\lambda)$, $C_3(\lambda)$ as in step **5** there. To prove the proposition, it is enough to make a choice of $\lambda>0$ such that for any fixed (ζ,ϵ) the set

$$\mathcal{M}_{i,j}(\zeta,\epsilon,\delta):=\{p\in\mathbb{Z}:X_{k_p}\in\Gamma_{\frac{\delta}{r_p}}(\beta_l),X_{k_{p-1}}\in\Gamma_{\frac{\delta}{r_p}}(\beta_l),|t-t_p|<\lambda\} \tag{3.129}$$

[52]The proof of Proposition 4.6 of [Wil20], which is analogous to but simpler than our Proposition 3.11, contained an error. This proof of Proposition 3.11 corrects that error.

has cardinality $|\mathcal{M}_{i,j}(\zeta,\epsilon,\delta)| \leq 1$.[53] For $\Omega := \Omega_{i,j}$ as in (3.126) we take

$$\lambda = \lambda_{i,j} = \frac{1}{3}\min\{|\Omega - (-1)|, |\Omega - 0|\} > 0. \tag{3.130}$$

Suppose $n > m$ are elements of $\mathcal{M}_{i,j}(\zeta,\epsilon,\delta)$. We have

$$\begin{aligned} X_{k_{n-1}} &= \zeta + k_{n-1}\frac{\beta_l}{\epsilon} = \zeta + (k_n - r_n)\frac{\beta_l}{\epsilon}, \\ \tilde{X}_{k_{n-1}} &= s\frac{\beta_l}{\epsilon} + (k_n - r_n)\frac{\beta_l}{\epsilon}, \end{aligned} \tag{3.131}$$

where $s\frac{\beta_l}{\epsilon}$ is the orthogonal projection of ζ on β_l for some $s \in \mathbb{R}$. Thus, we can write the "$t-$values" determined by $\tilde{X}_{k_{n-1}}$, $\tilde{X}_{k_{m-1}}$ as

$$t(n) = s + (k_n - r_n), \quad t(m) = s + (k_m - r_m). \tag{3.132}$$

The assumption that $n, m \in \mathcal{M}_{i,j}$ means

$$\begin{aligned} &(a)|s + (k_n - r_n) - r_n\Omega| < \lambda \text{ and } |s + (k_m - r_m) - r_m\Omega| < \lambda, \text{ so} \\ &(b)|(k_n - k_m) - (r_n - r_m) - (r_n - r_m)\Omega| < 2\lambda. \end{aligned} \tag{3.133}$$

2. The inequality (3.133)(b) shows that $r_n \neq r_m$, since otherwise $1 < \frac{2\lambda}{k_n - k_m}$, which is untrue. Thus, (3.133)(b) implies

$$\left|\left(\frac{k_n - k_m}{r_n - r_m} - 1\right) - \Omega\right| < \frac{2\lambda}{|r_n - r_m|} \leq 2\lambda. \tag{3.134}$$

If $r_n > r_m$, the quantity $\left(\frac{k_n - k_m}{r_n - r_m} - 1\right) > 0$ since $\frac{k_n - k_m}{r_n} \geq 1$, and so (3.134) is impossible since $|\Omega| \geq 3\lambda$. If $r_n < r_m$, the quantity $\left(\frac{k_n - k_m}{r_n - r_m} - 1\right) < -1$, and so (3.134) contradicts the fact that $|-1-\Omega| \geq 3\lambda$. Thus, $|\mathcal{M}_{i,j}(\zeta,\epsilon,\delta)| \leq 1$.

3. If $\mathcal{M}_{i,j}(\zeta,\epsilon,\delta)$ is nonempty, we denote its single element by $m = m^{i,j}(\zeta,\epsilon)$. Since $|t_p| \lesssim r_p$, the estimate (3.128) follows directly from (3.116) and (3.103).

□

Observe that $\Omega_{i,N} = -1$. This case, which is always a good one, is treated in the next proposition. This proposition is useful for counting large amplification factors (Proposition 4.3) and is also needed in the proof of Theorem 2.12.

Proposition 3.22. *Suppose $i \in \mathcal{O}$, $j = N \in \mathcal{I}$, so that $\Omega_{i,N} = -1$. There exist positive constants ϵ_0, δ_0 and a positive constant C_3 independent of $(\zeta,\epsilon,p) \in \Xi \times (0,\epsilon_0] \times \mathbb{Z}$ such that for any given $(\zeta,\epsilon,\delta) \in \Xi \times (0,\epsilon_0] \times (0,\delta_0]$, if $X_k \in \Gamma_{\frac{\delta}{|r|}}(\beta_l)$ and $X_{k-r} \in \Gamma_{\frac{\delta}{|r|}}(\beta_l)$, then*

$$|E_{i,N}(\epsilon,k,k-r;\beta_l)| \geq C_3\frac{|X_k|}{|r|} \text{ or } |E_{i,N}(\epsilon,k,k-r;\beta_l)| \geq C_3|X_{k-r}|. \tag{3.135}$$

[53] Recall that for a given (ζ,ϵ,p), $t = t(p)$ was defined in (3.103) by $\tilde{X}_{k_{p-1}} = t\frac{\beta_l}{\epsilon}$. Also, from (3.105) we have $t_p := r_p\Omega_{i,j}$.

Proof. Using notation similar to that in the proof of Proposition 3.10, we define t by $\tilde{X}_{k-r} = t\frac{\beta_l}{\epsilon}$. Since $\tilde{X}_k = (t+r)\frac{\beta_l}{\epsilon}$, we obtain

$$
\begin{gathered}
|\tilde{E}_{i,N}(\epsilon,k,k-r)| = \left|\omega_i\left((t+r)\frac{\beta_l}{\epsilon}\right) - r\omega_N\left(\frac{\beta_l}{\epsilon}\right) - \omega_N\left(t\frac{\beta_l}{\epsilon}\right)\right| = \\
\left|\omega_i\left((t+r)\frac{\beta_l}{\epsilon}\right) - \omega_N\left((t+r)\frac{\beta_l}{\epsilon}\right)\right| \sim \left|(t+r)\frac{\beta_l}{\epsilon}\right| \sim |\tilde{X}_k|.
\end{gathered} \tag{3.136}
$$

Since $|X_k| \sim |\tilde{X}_k|$ and

$$
\begin{gathered}
|\omega_i(X_k) - \omega_i(\tilde{X}_k)| \lesssim \frac{\delta}{|r|}|\tilde{X}_k|, \\
\left|\omega_N\left(X_k - r\frac{\beta_l}{\epsilon}\right) - \omega_N\left(\tilde{X}_k - r\frac{\beta_l}{\epsilon}\right)\right| \lesssim |X_k - \tilde{X}_k| \lesssim \frac{\delta}{|r|}|\tilde{X}_k|,
\end{gathered} \tag{3.137}
$$

the first alternative in (3.135) follows from (3.136) for δ small.

□

4 Cascade estimates

Our main concern in this section is to finish the proofs of Theorems 2.11 and 2.12. For this we must show how the iteration estimate of Proposition 3.18 can be used to prove useful energy estimates for the singular system (1.7). The first task is to develop an efficient procedure for managing the proliferation of terms that arise when the iteration estimate is iterated in the one-sided case.

4.1 One-sided cascade estimates and proof of Theorem 2.11

As in the Introduction we first consider the transformed singular problem (1.13) in the case where $F = 0$ and $G_k = 0$ for $k < 1$.

A more efficient way to obtain the essential information in (1.18), (1.19) or (1.20) is to consider the following "$\mathcal{G}_j$-cascades" corresponding to the cascades (1.19), (1.20):

$$
\begin{aligned}
&(a)[(\mathcal{G}_3)] \to [(\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_1)] \to [(\mathcal{G}_1)] \\
&(b)[(\mathcal{G}_5)] \to [(\mathcal{G}_4,\mathcal{G}_3,\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_3,\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_1)] \to \\
&\quad [(\mathcal{G}_3,\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_1),(\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_1),(\mathcal{G}_1),(\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_1),(\mathcal{G}_1),(\mathcal{G}_1)] \to \\
&\qquad [(\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_1),(\mathcal{G}_1),(\mathcal{G}_1),(\mathcal{G}_1),(\mathcal{G}_1)] \to [(\mathcal{G}_1)]
\end{aligned} \tag{4.1}
$$

The principle is slightly different here; for example, the $j-$th stage of (4.1)(b) records only the *new* $\mathcal{G}_p$ that appear in the $(j+1)$-st stage of (1.20). Consequently, the thirty-four $\mathcal{G}_p$ terms that appear in (4.1)(b) are the same as the thirty-four $\mathcal{G}_p$ terms that appear in the last stage of (1.20).

The "rule" for constructing a $\mathcal{G}_j$-cascade is that a term $\mathcal{G}_p$ in a given stage should give rise to the terms

$$
\mathcal{G}_p \to (\mathcal{G}_{p-1},\mathcal{G}_{p-2},\dots,\mathcal{G}_1,\mathcal{G}_{p-2},\dots,\mathcal{G}_1,\dots,\mathcal{G}_3,\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_1,\mathcal{G}_1). \tag{4.2}
$$

in the next stage. Terms with indices $j \leq 0$ are omitted, since we are assuming for now that $G_j = 0$ for $j \leq 0$. Thus, for example, $\mathcal{G}_5 \to (\mathcal{G}_4, \mathcal{G}_3, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1)$, while $\mathcal{G}_3 \to (\mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1)$ and $\mathcal{G}_2 \to (\mathcal{G}_1)$.

The parentheses in (4.1) allow us to track the "genealogy" of each $\mathcal{G}_j$. For example, we regard each of the $\mathcal{G}_j$ in the second group $(\mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1)$ that appears in the third stage of (4.1)(b) as a "descendant" of the second $\mathcal{G}_3$ that appears in the second stage. Similarly, the $\mathcal{G}_1$ that appears in the final stage of (4.1)(a) is a descendant of the $\mathcal{G}_2$ in the second stage. The number of arrows that precede the stage in which a given $\mathcal{G}_j$ lies tells us the number of factors of the form $\frac{C}{\gamma}\alpha_r\mathbb{D}(\epsilon, p, p-r)$ that should multiply that $\mathcal{G}_j$ in the final estimate; the particular choices of p and r appearing in those factors are determined by the genealogy of the given $\mathcal{G}_j$. For example, the term $[(\mathcal{G}_1)]$ in the final stage of (4.1)(a) should have two such factors attached, and indeed this term corresponds to the first term inside the brackets in the last line of (1.18). Keeping in mind the iteration estimate (1.17), one can easily reconstruct the complete estimate of V_3 starting just from (4.1)(a), and the same applies to any V_k. For example, the last $(\mathcal{G}_1)$ term appearing in the fourth stage of (4.1)(b) corresponds to the following term on the right in the final estimate of V_5:

$$\left|\left(\frac{C}{\gamma}\right)^3 \alpha_2\mathbb{D}(\epsilon,5,3)\alpha_1\mathbb{D}(\epsilon,3,2)\alpha_1\mathbb{D}(\epsilon,2,1)\mathcal{G}_1\right|_{L^2(\sigma,\eta)}. \tag{4.3}$$

Remark 4.1. *1) In the estimate of V_k, terms appear that involve a product of up to $k-1$ factors of the form $\frac{C}{\gamma}\alpha_r\mathbb{D}(\epsilon,p,p-r)$, where $2 \leq p \leq k$. A term involving three such factors in the estimate of V_5 is given by* (4.3)*. Observe that the product of $\mathbb{D}(\epsilon,p,p-r)$ factors in* (4.3) *is a special case of a finite product of the form*

$$\mathbb{D}(\epsilon,k_j,k_{j-1})\mathbb{D}(\epsilon,k_{j-1},k_{j-2})\mathbb{D}(\epsilon,k_{j-2},k_{j-3})\cdots \tag{4.4}$$

where the k_p that appear are elements of an admissible sequence $(k_j)_{j\in\mathbb{Z}}$. The results proved in section 3.5 will allow us to control these products by using the fact that they are always *either of the form* (4.4)*, or can be embedded in products of that form. For example, the product $\mathbb{D}(\epsilon,8,7)\mathbb{D}(\epsilon,7,5)\mathbb{D}(\epsilon,3,2)$ does not have this form, but can be embedded in*

$$\mathbb{D}(\epsilon,8,7)\mathbb{D}(\epsilon,7,5)\mathbb{D}(\epsilon,5,3)\mathbb{D}(\epsilon,3,2),$$

which has the right form. The large factors are counted in Proposition 4.3 below.

2) We will see that it is not necessary to keep track of the exact indices $(p,p-r)$ that appear in the individual factors, but only to keep track of the "step sizes", where r is the step size of the pair $(p,p-r)$, and of the various factors α_t contributed by the inner sum in the iteration estimate (see (4.12) *and* (4.18)*).*

Part (a) of Theorem 2.11 will be a direct consequence of Proposition 3.21 and the following proposition, whose proof occupies most of the rest of section 4.1.

Proposition 4.2. *Consider solutions $U(t,x,\theta)$ of the singular system* (1.7) *with forcing terms $F = \sum_{k\in\mathbb{Z}} F_k(t,x)e^{ik\theta}$, $G = \sum_{k\in\mathbb{Z}} G_k(t,x_1)e^{ik\theta}$ in $H^1(\Omega\times\mathbb{T})$, $H^1(\mathbb{R}^2\times\mathbb{T})$ respectively, under Assumptions 2.1, 2.3, 2.9. Assume the $N\times N$ matrices $\widehat{\mathcal{D}}(r)$ in* (2.23) *satisfy*

$$\widehat{\mathcal{D}}(r) = 0 \textit{ for } r \leq 0, \quad |\widehat{\mathcal{D}}(r)| \lesssim |r|^{-(M+3)} \textit{ for some } M \geq 2. \tag{4.5}$$

For $i \in \mathcal{O}$, $j \in \mathcal{I} \setminus \{N\}$ let [54]

$$\Omega_{i,j} := \frac{\omega_i(\beta_l) - \omega_N(\beta_l)}{\omega_j(\beta_l) - \omega_i(\beta_l)}. \tag{4.6}$$

Suppose there exist positive constants ϵ_0, δ_0 such that for $0 < \epsilon \leq \epsilon_0$, $0 < \delta \leq \delta_0$, $\zeta \in \Xi$ and any strictly increasing sequence of integers (k_p), there exist numbers $\mathbb{M}_{i,j} \geq 0$, $\lambda_{i,j} > 0$ independent of $(\zeta, \epsilon, \delta)$ and (k_p), such that the set

$$\mathcal{M}_{i,j}(\zeta, \epsilon, \delta) := \{p \in \mathbb{Z} : X_{k_p} \in \Gamma_{\frac{\delta}{r_p}}(\beta_l), X_{k_{p-1}} \in \Gamma_{\frac{\delta}{r_p}}(\beta_l), |t(p) - r_p \Omega_{i,j}| < \lambda_{i,j}\} \tag{4.7}$$

has finite cardinality $|\mathcal{M}_{i,j}(\zeta, \epsilon, \delta)| \leq \mathbb{M}_{i,j}$.[55] *Define the natural number $\mathbb{E}$ by*

$$\mathbb{E} = 2\left(|\Upsilon_0^+| - 1\right) + \sum_{i \in \mathcal{O}, j \in \mathcal{I} \setminus \{N\}} \mathbb{M}_{i,j}, \tag{4.8}$$

where we set $\sum \mathbb{M}_{i,j} = 0$ in case $\mathcal{I} = \{N\}$. Then there exist positive constants γ_0, K such that for $0 < \epsilon < \epsilon_0$ and $\gamma \geq \gamma_0$ the main energy estimate (2.28) holds with $\mathbb{E}$ given by (4.8).

The next proposition identifies the number $\mathbb{E}$ in (4.8) as an upper bound on the number of "large" amplification factors in products like (4.4).

Proposition 4.3 (Counting the large amplification factors)**.** *Let $(k_p)_{p \in \mathbb{Z}}$ be an admissible sequence and consider any finite product of the form (4.4), where the factors $\mathbb{D}(\epsilon, k_p, k_{p-1})(\zeta)$ are defined in Definition 3.16. Then for any given $(\epsilon, \zeta) \in (0, \epsilon_0] \times \Xi$, at most*

$$\mathbb{E} = 2(|\Upsilon_0^+| - 1) + \sum_{i \in \mathcal{O}, j \in \mathcal{I} \setminus \{N\}} \mathbb{M}_{i,j}$$

of the factors in that product are "large", that is, equal to $\frac{C_5|r|^3}{\epsilon\gamma}$. Here the $\mathbb{M}_{i,j}$ are as in Proposition 4.2. As (ϵ, ζ) varies, the particular indices p for which $\mathbb{D}(\epsilon, k_p, k_{p-1})(\zeta)$ is large can vary.

Proof. **1.** We will refer to cases $(I) - (III)$ as in (3.25) and Definition 3.15. For each $\beta \neq \beta_l$ the microlocal factor $D(\epsilon, k_p, k_{p-1}; \beta)(\zeta)$ can be large only in cases (I) or (II) when (3.18) fails. Proposition 3.11 shows that this can happen for at most two choices of $p \in \mathbb{Z}$. Thus, at most $2(|\Upsilon_0^+| - 1)$ of the factors $\mathbb{D}(\epsilon, k_p, k_{p-1})$ can be large due to largeness of $D(\epsilon, k_p, k_{p-1}; \beta)(\zeta)$ for some $\beta \neq \beta_l$.

2. A factor $D(\epsilon, k_p, k_{p-1}; \beta_l)(\zeta)$ can be large only if case (Ib) holds for some $(i,j) \in \mathcal{O} \times \mathcal{I}$. Step **6** of the proof of Proposition 3.10 shows that for a given pair $(i,j) \in \mathcal{O} \times (\mathcal{I} \setminus \{N\})$, there can be at most $\mathbb{M}_{i,j}$ indices p for which this happens. In addition, Proposition 3.22 (or Remark 3.19) shows that case (Ib) never holds for pairs (i, N), $i \in \mathcal{O}$. Thus, at most $\sum_{i \in \mathcal{O}, j \in \mathcal{I} \setminus \{N\}} \mathbb{M}_{i,j}$ distinct factors $\mathbb{D}(\epsilon, k_p, k_{p-1})(\zeta)$ can be large due to largeness of $D(\epsilon, k_p, k_{p-1}; \beta_l)(\zeta)$ for this reason.

3. Thus, at most $2(|\Upsilon_0^+| - 1) + \sum_{i \in \mathcal{O}, j \in \mathcal{I} \setminus \{N\}} \mathbb{M}_{i,j} = \mathbb{E}$ factors $\mathbb{D}(\epsilon, k_p, k_{p-1})(\zeta)$ in the given product can be large.

□

[54] Here the $\omega_j(\beta_l)$ are the distinct real eigenvalues of $\mathcal{A}(\beta_l)$; recall (2.8).

[55] Here as in (3.116) $t = t(p)$ is given by $\tilde{X}_{k_{p-1}} = t(p)\frac{\beta_l}{\epsilon}$, where $\tilde{X}_{k_{p-1}}$ is the orthogonal projection of $X_{k_{p-1}}$ on β_l.

4.1.1 Schematic representation of the V_k estimates

Observe that by (2.24) and Definition 3.16 we have for each (ζ,ϵ):[56]

$$\frac{C}{\gamma}|\alpha_r\mathbb{D}(\epsilon,p,p-r)(\zeta)| \lesssim \frac{C}{\gamma}\frac{1}{r^M} \text{ or } \frac{C}{\gamma}|\alpha_r\mathbb{D}(\epsilon,p,p-r)(\zeta)| \lesssim \frac{C}{\gamma}\frac{1}{\epsilon\gamma}\frac{1}{r^M}. \tag{4.9}$$

Let us define $\mathcal{D}_r(\zeta)$ to be the function of ζ:

$$\mathcal{D}_r(\zeta) = \begin{cases} \frac{C}{\gamma}\frac{1}{r^M}, \text{ if } \frac{C}{\gamma}|\alpha_r\mathbb{D}(\epsilon,p,p-r)(\zeta)| \lesssim \frac{C}{\gamma}\frac{1}{r^M} \\ \frac{C}{\gamma}\frac{1}{\epsilon\gamma}\frac{1}{r^M}, \text{ if not} \end{cases}. \tag{4.10}$$

We claim that we can represent the essential aspects of the estimate (1.18) of V_3 schematically by $V_3 \le \mathcal{G}_3^T$ and, more generally, represent the estimate of V_k by

$$V_k \le \mathcal{G}_k^T, \tag{4.11}$$

where the $\mathcal{G}_k^T$ are defined recursively by [57]

$$\begin{aligned}
&\mathcal{G}_1^T = \mathcal{G}_1\\
&\mathcal{G}_2^T = \mathcal{G}_2 + \mathcal{D}_1\mathcal{G}_1\\
&\mathcal{G}_3^T = \mathcal{G}_3 + \mathcal{D}_1\mathcal{G}_2^T + (\mathcal{D}_1+\mathcal{D}_2)\mathcal{G}_1\\
&\mathcal{G}_4^T = \mathcal{G}_4 + \mathcal{D}_1\mathcal{G}_3^T + (\mathcal{D}_1+\mathcal{D}_2)\mathcal{G}_2^T + (\alpha_2\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)\mathcal{G}_1\\
&\mathcal{G}_5^T = \mathcal{G}_5 + \mathcal{D}_1\mathcal{G}_4^T + (\mathcal{D}_1+\mathcal{D}_2)\mathcal{G}_3^T + (\alpha_2\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)\mathcal{G}_2^T + (\alpha_3\mathcal{D}_1+\alpha_2\mathcal{D}_2+\mathcal{D}_3+\mathcal{D}_4)\mathcal{G}_1\\
&\mathcal{G}_6^T = \mathcal{G}_6 + \mathcal{D}_1\mathcal{G}_5^T + (\mathcal{D}_1+\mathcal{D}_2)\mathcal{G}_4^T + (\alpha_2\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)\mathcal{G}_3^T +\\
&\qquad (\alpha_3\mathcal{D}_1+\alpha_2\mathcal{D}_2+\mathcal{D}_3+\mathcal{D}_4)\mathcal{G}_2^T + (\alpha_4\mathcal{D}_1+\alpha_3\mathcal{D}_2+\alpha_2\mathcal{D}_3+\mathcal{D}_4+\mathcal{D}_5)\mathcal{G}_1\\
&\dots
\end{aligned} \tag{4.12}$$

The factors α_t that appear in (4.12) come from the inner sum in the interaction estimate (3.48). Writing out $\mathcal{G}_3^T$ we obtain:

$$\mathcal{G}_3^T = \mathcal{G}_3 + \mathcal{D}_1\mathcal{G}_2 + \mathcal{D}_1^2\mathcal{G}_1 + \mathcal{D}_1\mathcal{G}_1 + \mathcal{D}_2\mathcal{G}_1. \tag{4.13}$$

The term $\mathcal{D}_1^2\mathcal{G}_1$, for example, "represents" the term $\left(\frac{C}{\gamma}\right)^2|\alpha_1\mathbb{D}(\epsilon,3,2)\alpha_1\mathbb{D}(\epsilon,2,1)\mathcal{G}_1|$ in (1.18), while $\mathcal{D}_2\mathcal{G}_1$ represents the term $\frac{C}{\gamma}|\alpha_2\mathbb{D}(\epsilon,3,1)\mathcal{G}_1|$. We will refer to (1.18) as "the proper estimate of V_3" and to $V_3 \le \mathcal{G}_3^T$ as "the schematic estimate of V_3". For every k there is a one-to-one correspondence between the terms of the proper estimate of V_k and those of the schematic estimate of V_k (after the $\mathcal{D}_r$ have been distributed as in (4.13)). We explain below how to transform schematic estimates, which can be stated with great concision, into (proper) estimates.

Consider a term on the right in the (proper) estimate of V_k, call it T, that consists of exactly k_0 factors of the form $\frac{C}{\gamma}\alpha_r\alpha_t\mathbb{D}(\epsilon,p,p-r)$ multiplying $\mathcal{G}_l$, for some $k_0 \le k-1$. Proposition 4.3 shows that

[56] In (4.9) we are asserting that there is a constant C such that either the first condition holds, *or* the first condition fails and the second condition holds.

[57] The superscript T in $\mathcal{G}_k^T$ is meant to indicate the "tree-like object" generated by $\mathcal{G}_k$.

for any fixed (ϵ, ζ) at most $\mathbb{E}$ of those factors fail to satisfy the first possibility in (4.9). Thus,

$$T \le \left(\frac{1}{\epsilon\gamma}\right)^{\mathbb{E}} \left(\frac{C}{\gamma}\right)^{k_0} \frac{|\alpha_{t_1}|}{r_1^M} \frac{|\alpha_{t_2}|}{r_2^M} \cdots \frac{|\alpha_{t_{k_0}}|}{r_{k_0}^M} |\mathcal{G}_l|, \tag{4.14}$$

where r_i is the step size of the $i-$th factor. The term T would be represented in the schematic estimate of V_k by

$$T = \alpha_{t_1} \mathcal{D}_{r_1} \alpha_{t_2} \mathcal{D}_{r_2} \cdots \alpha_{t_{k_0}} \mathcal{D}_{r_{k_0}} \mathcal{G}_l, \tag{4.15}$$

and we may represent (4.14) by[58]

$$T \le \left(\frac{1}{\epsilon\gamma}\right)^{\mathbb{E}} \alpha_{t_1} \mathcal{E}_{r_1} \alpha_{t_2} \mathcal{E}_{r_2} \cdots \alpha_{t_{k_0}} \mathcal{E}_{r_{k_0}} \mathcal{G}_l, \text{ where } \mathcal{E}_r := \frac{C}{\gamma} \frac{1}{r^M}. \tag{4.16}$$

Moreover, (4.16) implies the schematic estimate

$$\mathcal{G}_k^T \le \left(\frac{1}{\epsilon\gamma}\right)^{\mathbb{E}} \mathcal{H}_k^T, \tag{4.17}$$

where $\mathcal{H}_k^T$ is defined inductively by

$$\begin{aligned}
\mathcal{H}_1^T &= \mathcal{G}_1 \\
\mathcal{H}_2^T &= \mathcal{G}_2 + \mathcal{E}_1 \mathcal{G}_1 \\
\mathcal{H}_3^T &= \mathcal{G}_3 + \mathcal{E}_1 \mathcal{H}_2^T + (\mathcal{E}_1 + \mathcal{E}_2) \mathcal{G}_1 \\
\mathcal{H}_4^T &= \mathcal{G}_4 + \mathcal{E}_1 \mathcal{H}_3^T + (\mathcal{E}_1 + \mathcal{E}_2) \mathcal{H}_2^T + (\alpha_2 \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) \mathcal{G}_1 \\
\mathcal{H}_5^T &= \mathcal{G}_5 + \mathcal{E}_1 \mathcal{H}_4^T + (\mathcal{E}_1 + \mathcal{E}_2) \mathcal{H}_3^T + (\alpha_2 \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) \mathcal{H}_2^T + (\alpha_3 \mathcal{E}_1 + \alpha_2 \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4) \mathcal{G}_1 \\
\mathcal{H}_6^T &= \mathcal{G}_6 + \mathcal{E}_1 \mathcal{H}_5^T + (\mathcal{E}_1 + \mathcal{E}_2) \mathcal{H}_4^T + (\alpha_2 \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) \mathcal{H}_3^T + \\
&\qquad (\alpha_3 \mathcal{E}_1 + \alpha_2 \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4) \mathcal{H}_2^T + (\alpha_4 \mathcal{E}_1 + \alpha_3 \mathcal{E}_2 + \alpha_2 \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5) \mathcal{G}_1 \\
&\cdots
\end{aligned} \tag{4.18}$$

For any k we have

$$\begin{aligned}
\mathcal{H}_k^T &= \mathcal{G}_k + \mathcal{E}_1 \mathcal{H}_{k-1}^T + (\mathcal{E}_1 + \mathcal{E}_2) \mathcal{H}_{k-2}^T + (\alpha_2 \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) \mathcal{H}_{k-3}^T + \\
&\quad (\alpha_3 \mathcal{E}_1 + \alpha_2 \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4) \mathcal{H}_{k-4}^T + \cdots + (\alpha_{k-2} \mathcal{E}_1 + \alpha_{k-3} \mathcal{E}_2 + \cdots + + \alpha_2 \mathcal{E}_{k-3} + \mathcal{E}_{k-2} + \mathcal{E}_{k-1}) \mathcal{G}_1.
\end{aligned} \tag{4.19}$$

Remark 4.4. *For $j \le k$ the coefficient of $\mathcal{G}_j$ in $\mathcal{H}_k^T$ equals the coefficient of $\mathcal{G}_{j+1}$ in $\mathcal{H}_{k+1}^T$. To see this look, for example, at the coefficients of $\mathcal{G}_1$ on the outermost diagonal of* (4.18) *ending say, at row 5.*[59] *These are the same as the coefficients of $\mathcal{G}_2$ or $\mathcal{H}_2^T$ on the first subdiagonal starting at row 2 and ending at row 6, and these are the same as the coefficients of $\mathcal{G}_3$ or $\mathcal{H}_3^T$ on the second subdiagonal starting at row 3 and ending at row 7, etc.. All this remains true if the α_i in* (4.18) *are replaced by $|\alpha_i|$.*

[58] Unlike (4.15) or $V_k \le \mathcal{G}_k^T$, the schematic estimate (4.16) is very close to a proper estimate. To obtain a proper estimate we just replace $\mathcal{G}_l$ on the right by $|\mathcal{G}_l|_{L^2(\sigma,\eta)}$ and replace each α_{t_i} by $|\alpha_{t_i}|$.

[59] These coefficients are $1, \mathcal{E}_1, \mathcal{E}_1 + \mathcal{E}_2, \alpha_2 \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3, \alpha_3 \mathcal{E}_1 + \alpha_2 \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4$.

Letting $g_{j,p}$ denote the coefficient of $\mathcal{G}_j$ in $\mathcal{H}_p^T$ after *the replacement of all α_i by $|\alpha_i|$, we have, consequently, the relation*

$$g_{j,p} = g_{1,p-(j-1)}, \tag{4.20}$$

which in view of (4.17) *implies the (proper) estimate*

$$\|V_k\| \leq \left(\frac{1}{\epsilon\gamma}\right)^{\mathbb{E}} \sum_{j=1}^{k} |\mathcal{G}_j|_{L^2(\sigma,\eta)} g_{1,k-(j-1)}. \tag{4.21}$$

Proposition 4.5 (Estimate of $|(\|V_k\|)|_{\ell^2}$)**.** *Consider the transformed singular problem* (3.31) *under the hypotheses of Theorem 2.11, but assume $F = 0$ and $G_k = 0$ for $k < 1$. In particular, we assume the coefficients α_r in* (3.31) *satisfy $|\alpha_r| \lesssim |r|^{-(M+3)}$ for some $M \geq 2$. Let $\mathbb{E}$ be as in Proposition 4.3. There exist positive constants K, γ_0 such that for $\gamma > \gamma_0$ we have*

$$|(\|V_k\|)|_{\ell^2} \leq \frac{K}{(\epsilon\gamma)^{\mathbb{E}}} |(|\mathcal{G}_k|_{L^2(\sigma,\eta)})|_{\ell^2}. \tag{4.22}$$

We can take $\gamma_0 = CC_M D_M$, where $C_M = 2 + \sum_{i=2}^{\infty} |\alpha_i|$, $D_M = \sum_{r=1}^{\infty} \frac{1}{r^M}$, and C is as in (4.10)*.*

Proof. **1.** From (4.21) and Young's inequality we obtain

$$|(\|V_k\|)|_{\ell^2} \leq \left(\frac{1}{\epsilon\gamma}\right)^{\mathbb{E}} |(|\mathcal{G}_k|_{L^2(\sigma,\eta)})|_{\ell^2} |(g_{1,k})|_{\ell^1}. \tag{4.23}$$

2. From (4.18) for $k \geq 2$ we clearly have

$$\begin{aligned} g_{1,k} = {} & \mathcal{E}_1 g_{1,k-1} + (\mathcal{E}_1 + \mathcal{E}_2) g_{1,k-2} + (|\alpha_2|\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) g_{1,k-3} + \\ & (|\alpha_3|\mathcal{E}_1 + |\alpha_2|\mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4) g_{1,k-4} + \cdots + \\ & (|\alpha_{k-2}|\mathcal{E}_1 + |\alpha_{k-3}|\mathcal{E}_2 + \cdots + |\alpha_2|\mathcal{E}_{k-3} + \mathcal{E}_{k-2} + \mathcal{E}_{k-1}) g_{1,1}. \end{aligned} \tag{4.24}$$

3. To sum the $g_{1,k}$, we write:

$$\begin{aligned} & g_{1,1} = 1 \\ & g_{1,2} = 0 + \mathcal{E}_1 g_{1,1} \\ & g_{1,3} = 0 + (\mathcal{E}_1 + \mathcal{E}_2) g_{1,1} + \mathcal{E}_1 g_{1,2} \\ & g_{1,4} = 0 + (|\alpha_2|\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) g_{1,1} + (\mathcal{E}_1 + \mathcal{E}_2) g_{1,2} + \mathcal{E}_1 g_{1,3} \\ & g_{1,5} = 0 + (|\alpha_3|\mathcal{E}_1 + |\alpha_2|\mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4) g_{1,1} + (|\alpha_2|\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) g_{1,2} + (\mathcal{E}_1 + \mathcal{E}_2) g_{1,3} + \mathcal{E}_1 g_{1,4} \\ & \dots \end{aligned} \tag{4.25}$$

Letting

$$\begin{aligned} \mathbb{E}_M := {} & \mathcal{E}_1 + (\mathcal{E}_1 + \mathcal{E}_2) + (|\alpha_2|\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) + (|\alpha_3|\mathcal{E}_1 + |\alpha_2|\mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4) + \\ & (|\alpha_4|\mathcal{E}_1 + |\alpha_3|\mathcal{E}_2 + |\alpha_2|\mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5) + \dots \end{aligned} \tag{4.26}$$

and summing (4.25) "by columns", we obtain

$$S := \sum_{k=1}^{\infty} g_{1,k} = 1 + g_{1,1}\mathbb{E}_M + g_{1,2}\mathbb{E}_M + g_{1,3}\mathbb{E}_M + \cdots = 1 + \mathbb{E}_M S. \tag{4.27}$$

Resumming (4.26) we obtain

$$\mathbb{E}_M = \mathcal{E}_1(1+1+|\alpha_2|+|\alpha_3|+|\alpha_4|+\dots) + \mathcal{E}_2(1+1+|\alpha_2|+|\alpha_3|+|\alpha_4|+\dots) + \cdots . \tag{4.28}$$

Recalling that $\mathcal{E}_r = \frac{C}{\gamma}\frac{1}{r^M}$ and setting $C_M = 2 + \sum_{i=2}^{\infty}|\alpha_i|$, $D_M = \sum_{r=1}^{\infty}\frac{1}{r^M}$, we have

$$\mathbb{E}_M = \frac{C}{\gamma}C_M D_M. \tag{4.29}$$

Thus, for $\gamma > CC_M D_M$ the sum S is finite and

$$S = \frac{1}{1-\mathbb{E}_M} = 1 + \mathbb{E}_M + \mathbb{E}_M^2 + \dots . \tag{4.30}$$

$\square$

We can now complete the proofs Proposition 4.2 and of Theorem 2.11 in the case where $\mathcal{D}(\theta_{in}) = d(\theta_{in})M$; for the general case see Remark 4.8.

Conclusion of the proof of Proposition 4.2. In the case where F_k and G_k vanish for $k < 1$ we have the iteration estimate (3.97). Iterating this estimate leads to a proliferation of both F_l and G_l terms, but the new F_l terms can be managed just like the G_l terms in the proof of Proposition 4.5. In place of (4.22) we have for some K and $\gamma \geq \gamma_0$:

$$|(\|V_k\|)|_{\ell^2} \leq \frac{K}{(\epsilon\gamma)^{\mathbb{E}}}\left[|(|\mathcal{F}_k|_{L^2(x_2,\sigma,\eta)})|_{\ell^2} + |(|\mathcal{G}_k|_{L^2(\sigma,\eta)})|_{\ell^2}\right], \tag{4.31}$$

where $\mathcal{F}_k := \frac{\widehat{F}_k|X_k|}{\gamma^2}$, $\mathcal{G}_k = \frac{\widehat{G}_k|X_k|}{\gamma^{3/2}}$, and $\mathbb{E}$ is as in Proposition 4.3.

The estimate (4.31) clearly holds with the same proof when the forcing terms are

$$F^{N^*} := \sum_{k=N^*}^{\infty} F_k(t,x)e^{ik\theta} \text{ and } G^{N^*} := \sum_{k=N^*}^{\infty} G_k(t,x_1)e^{ik\theta} \text{ for } N^* \in \mathbb{Z}.$$

Given general periodic functions $F(t,x,\theta) \in H^1(t,x,\theta)$ and $G(t,x_1,\theta) \in H^1(t,x_1,\theta)$, we define F^{N^*} and G^{N^*} by truncation, and obtain (2.28) in the limit as $N^* \to -\infty$. Here we have used the fact that the constants γ_0, K appearing in (4.31) are independent of ϵ and N^*; recall Remark 3.20. The estimate (2.28) then follows from (3.47).

$\square$

Conclusion of the proof of Theorem 2.11. **Part (a).** Let $\mathcal{M}_{i,j}(\zeta,\epsilon,\delta)$ be as in (4.7), where the $\lambda_{i,j}$ are defined by

$$\lambda_{i,j} = \frac{1}{3}\min\{|\Omega_{i,j} - (-1)|, |\Omega_{i,j} - 0|\} > 0. \tag{4.32}$$

Proposition 3.21 and its proof show that

$$|\mathcal{M}_{i,j}(\zeta,\epsilon,\delta)| \leq 1 \text{ for all } (i,j) \in \mathcal{O} \times (\mathcal{I} \setminus \{N\}). \tag{4.33}$$

Thus, the hypotheses of Proposition 4.2 are satisfied with

$$\mathbb{M}_{i,j} = 1, i \in \mathcal{O}, j \in (\mathcal{I} \setminus \{N\}), \tag{4.34}$$

and the main estimate (2.28) holds with $\mathbb{E}$ given by (2.27).

Part (b). Let $\mathcal{P} = \{r \in \mathbb{N} : |\alpha_r| \neq 0\}$; so by assumption we have $|\mathcal{P}| = P$. For a given (ϵ, ζ) consider any finite product of amplification factors

$$\mathbb{D}(\epsilon, k_{p_1}, k_{p_1-1})(\zeta) \cdot \mathbb{D}(k_{p_2}, k_{p_2-1})(\zeta) \cdot \cdots \cdot \mathbb{D}(\epsilon, k_{p_{N^*}}, k_{p_{N^*}-1})(\zeta), \quad N^* \in \mathbb{N} \tag{4.35}$$

that might now appear in (a term on the right side of) the estimate of some V_k. Observe that for every $l \in \{1, \ldots, N^*\}$ the step size[60]

$$r_{p_l} = k_{p_l} - k_{p_l-1} \in \mathcal{P}. \tag{4.36}$$

For $(i,j) \in \mathcal{O} \times (\mathcal{I} \setminus \{N\})$ define the set

$$\mathcal{M}_{i,j}(\zeta, \epsilon, \delta; N^*) = \left\{ l \in \{1, \ldots, N^*\} : X_{k_{p_l}} \in \Gamma_{\frac{\delta}{r_{p_l}}}, X_{k_{p_l-1}} \in \Gamma_{\frac{\delta}{r_{p_l}}}, |t(p_l) - r_{p_l}\Omega_{i,j}| < \frac{1}{2} \right\}, \tag{4.37}$$

where $t(p_l)$ is given by $\tilde{X}_{k_{p_l-1}} = t(p_l)\frac{\beta_l}{\epsilon}$ (recall (3.103)). Since we have $|t(p_l) - t(p_m)| \geq 1$ if $l \neq m$, it follows that

$$|\mathcal{M}_{i,j}(\zeta, \epsilon, \delta; N^*)| \leq P. \tag{4.38}$$

One can now repeat the proof of Proposition 4.3 with $\mathbb{M}_{i,j} = P$ for $i \in \mathcal{O}, j \in \mathcal{I} \setminus \{N\}$ to obtain the upper bound $\mathbb{E} = P|\mathcal{O}|(|\mathcal{I}| - 1) + 2(|\Upsilon_0^+| - 1)$ for the number of large factors in the product (4.35).[61]

□

4.2 An effect of resonances

Consider again the system (2.1) under assumptions 2.1, 2.3, 2.9. For N and $\beta_l \in \Upsilon_+^0$ as in (2.2), suppose $j, N \in \mathcal{I}$ with $j \neq N$ and $i \in \mathcal{O}$. We say that the associated characteristic phases (ϕ_j, ϕ_N, ϕ_i) exhibit a *resonance* if there exist $p, q \in \mathbb{Z} \setminus 0$ such that[62]

$$p\phi_j + q\phi_N = (p+q)\phi_i \Leftrightarrow p\omega_j(\beta_l) + q\omega_N(\beta_l) = (p+q)\omega_i(\beta_l) \Leftrightarrow \frac{p}{q} = \frac{\omega_i(\beta_l) - \omega_N(\beta_l)}{\omega_j(\beta_l) - \omega_i(\beta_l)} = \Omega_{i,j}, \tag{4.39}$$

for $\Omega_{i,j}$ as in (2.25).

For a given $C_3 > 0$ and a given admissible sequence (k_p), we recall the definition of the bad set $M_{i,j}(\epsilon, \zeta, \delta; C_3)$ from Proposition 3.10. The integer $p \in M_{i,j}(\epsilon, \zeta, \delta; C_3)$ if and only if both X_{k_p}, $X_{k_{p-1}}$ lie in $\Gamma_{\frac{\delta}{|r_p|}}(\beta_l)$ and

$$|E_{i,j}(\epsilon, k_p, k_{p-1})| \geq C_3 \frac{|X_{k_p}|}{|r_p|} \text{ or } |E_{i,j}(\epsilon, k_p, k_{p-1})| \geq C_3 |X_{k_{p-1}}| \tag{4.40}$$

[60] As always it can happen that $r_p = r_q$ for some $p \neq q$ in such a product.

[61] Here it does not help to apply Proposition 4.3 as stated, since we must now take advantage of the fact that the number of distinct possible step sizes is $\leq P$.

[62] Here, recall $\phi_j(t,x) = \beta_l \cdot (t, x_1) + \omega_j(\beta_l)x_2$.

fails to hold. Propositions 3.10 and 3.21 showed that when $\Omega_{i,j} \in (-1, 0)$, one can choose C_3 so that $|M_{i,j}(\epsilon, \zeta, \delta; C_3)| \leq 1$, and this was an essential step in the proof of Theorem 2.11(a). The next proposition shows that for certain resonances, there exist admissible sequences (k_p) and sets of ζ of large measure for which the set $M_{i,j}(\epsilon, \zeta, \delta; C_3)$ is infinite no matter how small $C_3 > 0$ is taken.

Proposition 4.6. *a) For any admissible sequence* (k_p)*, let* $\tilde{E}_{i,j}(\epsilon, k_p, k_{p-1})$ *be given by* (3.102) *with* $\beta = \beta_l$*. Let* $0 < \alpha < 1$*, and suppose there is a resonance such that* $\Omega_{i,j}$ *as in* (4.39) *satisfies*

$$\Omega_{i,j} = \frac{p}{q} \in \mathbb{Q} \cap (-\infty, -1) \text{ or } \Omega_{i,j} \in \mathbb{Q} \cap (0, \infty). \tag{4.41}$$

Then one can construct admissible sequences (k_p) *such that for all* $\zeta \in \Xi$ *with* $|\zeta| \leq \epsilon^{\alpha-1}$ *we have*

$$|\tilde{E}_{i,j}(\epsilon, k_p, k_{p-1})| \leq C(\beta_l)\epsilon^{\alpha-1} \text{ for infinitely many } p. \tag{4.42}$$

b) For each admissible sequence (k_p) *constructed in part (a) and for* $\epsilon > 0$ *and* $\delta > 0$ *small enough, there are subsets of* Ξ *of large measure (* $|\zeta| \leq \epsilon^{\alpha-1}$*) for which it is impossible to choose a constant* $C_3 > 0$ *independent of* (ϵ, ζ, p) *such that* $M_{i,j}(\epsilon, \zeta, \delta; C_3)$ *is finite.*

c) If $\Omega_{i,j} \in \mathbb{Q} \cap (-1, 0)$*, then for any admissible sequence* (k_p) *and for* $C_3 > 0$ *as chosen in Proposition 3.21, we have* $|M_{i,j}(\epsilon, \zeta, \delta; C_3)| \leq 1$ *for* $\epsilon > 0$ *and* $\delta > 0$ *both small enough.*

Proof. **1. Part a.** Consider the case $\Omega_{i,j} = \frac{p}{q} \in \mathbb{Q} \cap (0, \infty)$; the other case of (4.41) is treated similarly. We may take $p, q \in \mathbb{N}$. Let $n \in \mathbb{N}$ and set

$$k(n) = n(p + q), r(n) := nq, \text{ so } k(n) - r(n) = np. \tag{4.43}$$

We have $X_{k(n)-r(n)} = \zeta + (k(n) - r(n))\frac{\beta_l}{\epsilon}$, so we may write $\tilde{X}_{k(n)-r(n)} = s\frac{\beta_l}{\epsilon} + (k(n) - r(n))\frac{\beta_l}{\epsilon}$, where $s\frac{\beta_l}{\epsilon}$ is the orthogonal projection of ζ on β_l. Setting $t = s + k(n) - r(n)$, we obtain as in (3.106):

$$\begin{aligned}\tilde{E}_{i,j}(\epsilon, k(n), k(n) - r(n))(\zeta) &= \frac{t - r(n)\Omega_{i,j}}{\epsilon} C(\beta_l) = \\ &\frac{s + k(n) - r(n) - r(n)\Omega_{i,j}}{\epsilon} C(\beta_l) = C(\beta_l)\frac{s}{\epsilon}.\end{aligned} \tag{4.44}$$

So if $|\zeta| \leq \epsilon^{\alpha-1}$, it follows that $\frac{|s|}{\epsilon} \leq \epsilon^{\alpha-1}$, and thus

$$|\tilde{E}_{i,j}(\epsilon, k(n), k(n) - r(n))(\zeta)| \leq C(\beta_l)\epsilon^{\alpha-1}. \tag{4.45}$$

Now choose $0 < n_1 < n_2 < n_3 < \dots$ such that

$$k(n_1) - r(n_1) < k(n_1) < k(n_2) - r(n_2) < k(n_2) < k(n_3) - r(n_3) < k(n_3) < \dots, \tag{4.46}$$

and relabel the respective elements of (4.46) as $k_1 < k_2 < k_3 < k_4 < \dots$. Then (4.45) implies that if $|\zeta| \leq \epsilon^{\alpha-1}$, we have

$$|\tilde{E}_{i,j}(\epsilon, k_{2m}, k_{2m-1})(\zeta)| \leq C(\beta_l)\epsilon^{\alpha-1} \text{ for all } m \in \mathbb{N}. \tag{4.47}$$

Part b. Let (k_p) be the admissible sequence constructed at the end of step **1**, so

$$k_{2m} = k(n_m) = n_m(p + q) \text{ and } r_{2m} = n_m q. \tag{4.48}$$

Suppose $|\zeta| \leq \epsilon^{\alpha-1}$ and fix *any* $C_3 > 0$ independent of (ϵ, ζ, m). We claim that for $m \geq 2$, the index $2m \in M_{i,j}(\epsilon, \zeta, \delta; C_3)$ for δ small enough and $0 < \epsilon \leq \epsilon_0(\delta)$, provided $\epsilon_0(\delta)$ is small enough.

Observe that for a given $\delta > 0$ and $\epsilon_0(\delta)$ small enough, the vectors $X_{k_{2m}} = \zeta + n_m(p+q)\frac{\beta_l}{\epsilon}$ and $X_{k_{2m-1}} = \zeta + n_m p\frac{\beta_l}{\epsilon}$ both lie in $\Gamma_{\frac{\delta}{r_{2m}}}(\beta_l) = \Gamma_{\frac{\delta}{n_m q}}(\beta_l)$ for $0 < \epsilon \leq \epsilon_0(\delta)$, and we have [63]

$$\frac{p+q}{q\epsilon} \sim \frac{|X_{k_{2m}}|}{r_{2m}} \leq |X_{k_{2m-1}}| \sim \frac{n_m p}{\epsilon} \text{ for } m \geq 2. \tag{4.49}$$

Let $m \geq 2$ and suppose that

$$|E_{i,j}(\epsilon, k_{2m}, k_{2m-1})| \geq C_3\frac{|X_{k_{2m}}|}{r_{2m}}. \tag{4.50}$$

From (3.110) and (3.111) we have

$$|E_{i,j}(\epsilon, k_{2m}, k_{2m-1}) - \tilde{E}_{i,j}(\epsilon, k_{2m}, k_{2m-1})| \lesssim \frac{\delta}{r_{2m}}|X_{k_{2m}}| + \frac{\delta}{r_{2m}}|X_{k_{2m-1}}| \lesssim \frac{\delta}{r_{2m}}|X_{k_{2m}}|. \tag{4.51}$$

Then (4.50) and (4.51) imply that for δ small enough and $0 < \epsilon \leq \epsilon_0(\delta)$,

$$|\tilde{E}_{i,j}(\epsilon, k_{2m}, k_{2m-1})| \geq \frac{C_3}{2}\frac{|X_{k_{2m}}|}{r_{2m}} \gtrsim \frac{1}{\epsilon}, \tag{4.52}$$

but this contradicts (4.47), and so (4.50) fails. From (4.49) we see then that (4.40) fails for $p = 2m$, $m = 2, 3, \dots$, establishing the claim.

3. Part (c) follows immediately from Proposition 3.21.

$\square$

4.3 The two-sided case and proof of Theorem 2.12

We first state a simple general result for problems with two-sided cascades under a boundedness assumption on the factors $\mathbb{D}(\epsilon, p, p-r)$. Recall the iteration estimate (3.48) for the singular transformed problem (3.31):

$$\|V_k\| \leq \frac{C}{\gamma}\sum_{r\in\mathbb{Z}\setminus 0}\sum_{t\in\mathbb{Z}}\|\alpha_r\alpha_t\mathbb{D}(\epsilon, k, k-r)V_{k-r-t}\| + C|\mathcal{F}_k|_{L^2(x_2,\sigma,\eta)} + C|\mathcal{G}_k|_{L^2(\sigma,\eta)},\ k \in \mathbb{Z}, \tag{4.53}$$

where we have set $\mathcal{F}_k := \frac{\widehat{F}_k|X_k|}{\gamma^2}$, $\mathcal{G}_k = \frac{\widehat{G}_k|X_k|}{\gamma^{3/2}}$.

Proposition 4.7. *Assume the coefficients α_r in (3.31) satisfy $|\alpha_r| \lesssim |r|^{-(M+1)}$ for some $M \geq 2$. Suppose there exist positive constants ϵ_0 and C such that for all $(\epsilon, \zeta, k, r) \in (0, \epsilon_0] \times \Xi \times \mathbb{Z} \times (\mathbb{Z} \setminus 0)$ we have*

$$|\mathbb{D}(\epsilon, k, k-r)(\zeta)| \leq C|r|. \tag{4.54}$$

Then there exist positive constants K, γ_0 such that for $\gamma > \gamma_0$ we have

$$|(\|V_k\|)|_{\ell^2} \leq K\left[|(|\mathcal{F}_k|_{L^2(x_2,\sigma,\eta)})|_{\ell^2} + |(|\mathcal{G}_k|_{L^2(\sigma,\eta)})|_{\ell^2}\right]. \tag{4.55}$$

[63]We remark that $\epsilon_0(\delta)$ does not depend on $r_{2m} = n_m q$, because of the factor n_m multiplying $\frac{\beta_l}{\epsilon}$ in the expressions for $X_{k_{2m}}$ and $X_{k_{2m-1}}$.

Proof. Letting $\beta_r := \alpha_r C|r|$, we have

$$\begin{aligned}(4.56)\qquad &\sum_{r\in\mathbb{Z}\setminus 0}\sum_{t\in\mathbb{Z}}\|\alpha_r\alpha_t\mathbb{D}(\epsilon,k,k-r)V_{k-r-t}\| \leq \sum_{r\in\mathbb{Z}\setminus 0}\sum_{t\in\mathbb{Z}}\|\beta_r\alpha_t V_{k-r-t}\| = \\ &\sum_s\left(\sum_{r+t=s}|\beta_r\alpha_t|\right)\|V_{k-s}\| := \sum_s \gamma_s\|V_{k-s}\|.\end{aligned}$$

Applying Young's inequality gives

$$\left|\left(\sum_s \gamma_s\|V_{k-s}\|\right)\right|_{\ell^2(k)} \leq |(\|V_k\|)|_{\ell^2}|(\gamma_s)|_{\ell^1}. \tag{4.57}$$

Since $\gamma_s = \sum_r |\beta_r||\alpha_{s-r}|$, applying Young's inequality again we obtain

$$|(\gamma_s)|_{\ell^1} \leq |(\beta_r)|_{\ell^1}|(\alpha_t)|_{\ell^1} := K_1. \tag{4.58}$$

Thus, the ℓ^2 norm of the right side of (4.53) is $\lesssim \frac{K_1}{\gamma}|(\|V_k\|)|_{\ell^2} + |(|\mathcal{F}_k|_{L^2(x_2,\sigma,\eta)})|_{\ell^2} + |(|\mathcal{G}_k|_{L^2(\sigma,\eta)})|_{\ell^2}$, and the result follows by taking γ_0 large enough.

∎

We can now finish the proof of Theorem 2.12 for the case where $\mathcal{D}(\theta_{in}) = d(\theta_{in})M$; for the general case see Remark 4.8.

Conclusion of the proof of Theorem 2.12. Since $\Upsilon_0^+ = \{\beta_l\}$ now, Proposition 3.18 and Remark 3.17 show that the estimate (4.53) holds with the definition of $\mathbb{D}(\epsilon,k,k-r)$ modified as in that remark (put $|r|$ in place of r^2 in (3.29)). Using Definition 3.16, we see that a factor $\mathbb{D}(\epsilon,k,k-r)(\zeta)$ can take the value $\frac{C_5|r|}{\epsilon\gamma}$ only if $D(\epsilon,k,k-r;\beta_l)(\zeta) = \frac{C_5|r|}{\epsilon\gamma}$, but Proposition 3.22 implies that this cannot happen since now $\mathcal{I} = \{N\}$. Thus, all factors occurring in the iteration estimate satisfy (4.54) with $C = C_5$. Application of Proposition 4.7 then yields the result. ∎

Remark 4.8. *[Reduction to the case $\mathcal{D}(\theta_{in}) = d(\theta_{in})M$.] Consider first the reduction in the case of Theorem 2.11. Writing as in* (1.8)

$$\mathcal{D}(\theta_{in}) = \sum_{i,j=1}^N d_{i,j}(\theta_{in})M_{i,j},\ \textit{where}\ d_{i,j}(\theta_{in}) = \sum_{r\in\mathbb{Z}\setminus 0}\alpha_r^{i,j}e^{ir\theta_{in}}, \tag{4.59}$$

we see that the transform of the singular problem (1.7) *is just like* (1.13)*, except that the sum on the right is replaced by*

$$\sum_{i,j=1}^N\sum_{r\in\mathbb{Z}\setminus 0}\alpha_r^{i,j}e^{ir\frac{\omega_N(\beta_l)}{\epsilon}x_2}B_2^{-1}M_{i,j}V_{k-r}. \tag{4.60}$$

The assumption (2.24) *implies*

$$\alpha_r^{i,j} = 0\ \textit{for}\ r\leq 0,\ |\alpha_r^{i,j}| \leq Ar^{-(M+3)}\ \textit{for}\ r\geq 1 \tag{4.61}$$

for some $M \geq 2$ and $A > 0$ (both) independent of (i,j). The solution formulas (3.38), (3.39) *and* (3.41), (3.42) *change in the obvious way when the replacement* (4.60) *is made. The proof of the iteration estimate* (3.48) *was based on an analysis of the individual terms (each associated to a particular choice of r) in the solution formulas. That analysis can be repeated for the new solution formulas to yield an iteration estimate of exactly the same form* (3.48)*, but with C replaced by N^2C and α_r redefined as*

$$\alpha_r = 0 \text{ for } r < 0, \quad \alpha_0 = 1, \quad \alpha_r = Ar^{-(M+3)} \text{ for } r \geq 1. \tag{4.62}$$

The cascade estimates leading to the proof of Theorem 2.11 for general $\mathcal{D}$ can be carried out exactly as before using the new iteration estimate.

The reduction in the case of Theorem 2.12 is carried out in the same way.

5 Multiple amplification and optimality of the estimates

In this section we give the details of Example 2.14; that is, we construct and rigorously justify geometric optics solutions to a 3×3, strictly hyperbolic WR problem of the form (2.35) on $\Omega_T = (-\infty, T] \times \{(x_1, x_2) : x_2 \geq 0\}$:

$$\begin{aligned}
&L(\partial)u + e^{i\frac{\phi_3}{\epsilon}}Mu := \partial_t u + B_1\partial_{x_1}u + B_2\partial_{x_2}u + e^{i\frac{\phi_3}{\epsilon}}Mu = 0 \text{ in } x_2 > 0\\
&Bu = \epsilon G(t, x_1, \frac{\phi_0}{\epsilon}) := \epsilon g_{-2}(t, x_1)e^{-i\frac{2\phi_0}{\epsilon}} \text{ on } x_2 = 0\\
&u = 0 \text{ in } t < 0,
\end{aligned} \tag{5.1}$$

which exhibit instantaneous double amplification. The main new element in the presentation of Example 2.14 is the construction of the approximate solution, which occupies most of the rest of this section. The rigorous justification of the approximate solution, given in section 5.6, is based on the L^2-estimate of Theorem 2.11(b) together with a higher derivative estimate proved as in section 6 of [Wil20].

Notations 5.1. *In previous sections we used $\theta \in \mathbb{R}$ as a place holder for $\frac{\phi_0}{\epsilon}$. In this section we use $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ as a placeholder for $\frac{\Phi}{\epsilon}$, $\Phi = (\phi_1, \phi_2, \phi_3)$, and we use θ_0 as a placeholder for $\frac{\phi_0}{\epsilon}$.*

Remark 5.2. *A 3×3 problem satisfying the conditions stated in Example 2.14 can be exhibited by choosing $(L(\partial), B)$ as in Appendix B of [CGW14]. There the operator $L(\partial)$ is obtained by linearizing the compressible 2D-Euler system at a given specific volume $v > 0$ and subsonic incoming velocity $(0, u)$, $0 < u < c$, where c is the sound speed. The boundary matrix B is chosen there so that $\Upsilon_0 = \{\beta_l, -\beta_l\}$, where $\beta_l = (c, 1)/|(c, 1)|$. The phases ϕ_j, $j = 1, \dots 3$ exhibit the single resonance*

$$-2\phi_2 + \phi_3 = -\phi_1, \tag{5.2}$$

provided the Mach number $\frac{u}{c}$ is chosen to be $\frac{1}{\sqrt{3}}$. The matrix M in (5.1) *is chosen to satisfy*[64]

$$Mr_3 = 0, \text{ where } \operatorname{Ker} L(d\phi_3) = \operatorname{span}\{r_3\}. \tag{5.3}$$

[64] This condition, which substantially simplifies the construction of profiles, may be inessential.

5.1 Tools for constructing approximate solutions

We recall here some useful results and introduce some notation. We set

$$(5.4)\qquad \begin{aligned} &L(\partial) = \partial_t + B_1\partial_{x_1} + B_2\partial_{x_2}, \quad L(\sigma,\eta,\xi) = \sigma I + B_1\eta + B_2\xi \\ &\mathcal{L}(\partial_\theta) = \sum_{m=1}^{3} L(d\phi_m)\partial_{\theta_m}, \quad \phi_m(t,x) = \beta_l\cdot(t,x_1) + \omega_m(\beta_l)x_2. \end{aligned}$$

Let $\mathcal{A}(\beta_l)$ be the matrix

$$(5.5)\qquad \mathcal{A}(\beta_l) = -(A_0\sigma_l + A_1\eta_l), \text{ where } A_0 = B_2^{-1}, A_1 = B_2^{-1}B_1.$$

The matrix $\mathcal{A}(\beta_l)$ is diagonalizable with eigenvalues $\omega_m(\beta_l) = \omega_m$, $m = 1,\dots,3$, and the eigenspace of $\mathcal{A}(\beta_l)$ for ω_m coincides with the kernel of $L(d\phi_m)$.

Lemma 5.3. *[CG10] The (extended) stable subspace $\mathbb{E}^s(\beta_l)$ (recall Prop. 2.7) admits the decomposition*

$$(5.6)\qquad \mathbb{E}^s(\beta_l) = \oplus_{m\in\mathcal{I}} \operatorname{Ker} L(d\phi_m)\,,$$

and each vector space in the decomposition (5.6) *is of real type (that is, it admits a basis of real vectors).*

Lemma 5.4. *[CG10] The following decompositions hold*

$$(5.7)\qquad \mathbb{C}^3 = \oplus_{m=1}^{3} \operatorname{Ker} L(d\phi_m) = \oplus_{m=1}^{3} B_2 \operatorname{Ker} L(d\phi_m)\,,$$

and each vector space in the decompositions (5.7) *is of real type.*

We let P_m, respectively, Q_m, $m = 1,2,3$, denote the projectors associated with the first, respectively, second decomposition in (5.7). *For each m there holds* $\operatorname{Im} L(d\phi_m) = \operatorname{Ker} Q_m$.

Using Lemma 5.4, we may introduce the partial inverse R_m of $L(d\phi_m)$, which is uniquely determined by the relations

$$\forall\, m = 1,2,3\,, \quad R_m\, L(d\phi_m) = I - P_m\,, \quad L(d\phi_m)R_m = I - Q_m, \quad P_m\, R_m = 0\,, \quad R_m\, Q_m = 0\,.$$

In the case of our strictly hyperbolic system (2.35), we choose for each m a real vector r_m that spans $\operatorname{Ker} L(d\phi_m)$. We also choose real row vectors ℓ_m, that satisfy

$$\forall\, m = 1,\dots,3\,, \quad \ell_m\, L(d\phi_m) = 0\,,$$

together with the normalization $\ell_m\, B_2\, r_{m'} = \delta_{mm'}$. With this choice, the partial inverse R_m and the projectors P_m, Q_m are given by[65]

$$\forall\, X \in \mathbb{C}^3\,, \quad R_m\, X = \sum_{m'\neq m} \frac{\ell_{m'}\, X}{\omega_m - \omega_{m'}}\, r_{m'} \quad P_mX = (\ell_m B_2 X)r_m, \quad Q_mX = (\ell_m X)B_2 r_m.$$

[65] To see this write $\mathcal{A}(\beta_l) = \sum_m \omega_m P_m$, which implies $L(d\phi_m) = \sum_{k\neq m}(\omega_m - \omega_k)B_2P_k$, and observe that $R_m = \sum_{k\neq m} \frac{P_k B_2^{-1}}{\omega_m - \omega_k}$.

In the analysis of the profile equations we use projection operators E_Q, E_P defined on $H^\infty:=H^\infty(\Omega_T\times \mathbb{T}^3)$ and a partial inverse R of $\mathcal{L}(\partial_\theta)$ defined on functions in a certain subspace of H^∞.We have

$$E_P = E_0 + \sum_{m=1}^{3} E_{P_m},\ E_{P_{in}} = E_{P_2} + E_{P_3}, \quad E_{P_{out}} = E_{P_1}, \tag{5.8}$$

and similarly expand E_Q, replacing P_m by Q_m. In (5.8) E_{P_0} picks out the mean and E_{P_m} picks out pure θ_m modes and then projects with P_m. More precisely, writing

$$U(t,x,\theta) = \underline{U}(t,x) + U^*(t,x,\theta) = \underline{U} + \sum_{m=1}^{3} U^m(t,x,\theta_m) + U^{nc}(t,x,\theta_1,\theta_2,\theta_3), \tag{5.9}$$

where each U^m has pure θ_m oscillations and mean zero and U^{nc} is obtained by retaining only noncharacteristic modes in the Fourier series of U, we have

$$\begin{aligned} &E_0 U = \underline{U},\ \ E_{P_m} U = P_m U^m(t,x,\theta_m),\ m=1,2,3 \\ &(I - E_P)U = \sum_{m=1}^{3} (I - P_m)U^m + U^{nc}. \end{aligned} \tag{5.10}$$

Thus, along with (5.9) we can decompose U as

$$U(t,x,\theta) = \underline{U} + E_{P_1}U + E_{P_{in}}U + (I - E_P)U, \tag{5.11}$$

and we have the obvious analogue of (5.11) for E_Q.

The definition of the partial inverse R of $\mathcal{L}(\partial_\theta)$ generally requires that a small divisor assumption is satisfied [JMR93]. In order to state the assumption relevant for Example 2.14 we first define the following subspace of H^∞.

Definition 5.5. *Define $\mathcal{N} := \{(k,l) \in (\mathbb{Z}\setminus 0)\times(\mathbb{Z}\setminus 0) : k \neq -2l,\ k \geq -2,\ l \geq -2\}$. Let $\mathcal{H}^\infty$ be the subspace of $H^\infty(t,x,\theta)$ given by*

$$\mathcal{H}^\infty := \{U \in H^\infty : U^{nc} = \sum_{(k,l)\in\mathcal{N}} c_{k,l}(t,x)e^{i(k\theta_2 + l\theta_3)}\}. \tag{5.12}$$

We make the following small divisor assumption for the system (5.1).

Assumption 5.6. *There exist constants $C > 0$ and $a \in \mathbb{R}$ such that for all $(k,l) \in \mathcal{N}$ we have*

$$|\det L(kd\phi_2 + ld\phi_3)| \geq C|(k,l)|^a. \tag{5.13}$$

Writing $U \in \mathcal{H}^\infty$ as in (5.9) with U^{nc} as in (5.12), we define

$$\begin{aligned} &R(\underline{U}) = 0 \\ &R(U^m) = \partial_{\theta_m}^{-1} R_m U^m \\ &R(U^{nc}) = \mathcal{L}(\partial_\theta)^{-1} U^{nc}. \end{aligned} \tag{5.14}$$

Here $\partial_{\theta_m}^{-1} R_m U^m$ denotes the unique mean zero primitive in θ_m of $R_m U^m$ and

$$\mathcal{L}(\partial_\theta)^{-1}U^{nc} = \sum_{(k,l)\in\mathcal{N}} L(ikd\phi_2 + ild\phi_3)^{-1} c_{k,l}(t,x) e^{i(k\theta_2+l\theta_3)}. \tag{5.15}$$

It is clear that as a consequence of the small divisor assumption, we have $R : \mathcal{H}^\infty \to \mathcal{H}^\infty$.

As operators on $\mathcal{H}^\infty$, the operators $\mathcal{L}(\partial_\theta)$, E_P, E_Q, and R are easily seen to satisfy

$$\begin{aligned}
&a) E_Q \mathcal{L}(\partial_\theta) = \mathcal{L}(\partial_\theta) E_P = 0,\\
&b) R\mathcal{L}(\partial_\theta) = I - E_P, \ \ \mathcal{L}(\partial_\theta) R = I - E_Q\\
&c) E_P R = R E_Q = 0.
\end{aligned} \tag{5.16}$$

Remark 5.7. *When the system* (5.1) *is chosen as in Remark 5.2, we will see that the profiles* $U(t,x,\theta)$ *that arise in the construction all lie in* $\mathcal{H}^\infty$. *Moreover, one can use the explicit formula for* $\det L(kd\phi_2 + ld\phi_3)$ *given just before Lemma A.1 in [CW17] to check that Assumption 5.6 holds with* $a = 0$ *when* $(L(\partial), B)$ *is chosen as in Remark 5.2.*[66]

5.2 Profile equations

We construct approximate solutions to the system (5.1) of the form

$$u_a^\epsilon(t,x) = \sum_{k=-1}^{J} \epsilon^k U_k\left(t, x, \frac{\Phi}{\epsilon}\right), \tag{5.17}$$

where the profiles U_k lie in $\mathcal{H}^\infty$. Plugging the ansatz (5.17) into the system (5.1) and setting coefficients of different powers of ϵ equal to zero, we obtain interior equations

$$\begin{aligned}
&a)\mathcal{L}(\partial_\theta)U_{-1} = 0\\
&b)\mathcal{L}(\partial_\theta)U_0 + L(\partial)U_{-1} + e^{i\theta_3} M U_{-1} = 0\\
&c)\mathcal{L}(\partial_\theta)U_1 + L(\partial)U_0 + e^{i\theta_3} M U_0 = 0\\
&d)\mathcal{L}(\partial_\theta)U_j + L(\partial)U_{j-1} + e^{i\theta_3} M U_{j-1} = 0, j \geq 2
\end{aligned} \tag{5.18}$$

and boundary equations

$$\begin{aligned}
&a) BU_{-1} = 0\\
&b) BU_0 = 0\\
&c) BU_1 = G(t, x_1, \theta_0) = g_{-2}(t, x_1) e^{-i2\theta_0}\\
&d) BU_j = 0, j \geq 2.
\end{aligned} \tag{5.19}$$

Remark 5.8. *The left and right sides of boundary equations are always evaluated at* $x_2 = 0$, $\theta_1 = \theta_2 = \theta_3 = \theta_0$.

[66] The phases we label as ϕ_2, ϕ_3 are labeled as ϕ_1, ϕ_3 in [CW17].

The vector space $E^s(\beta_l)$ is spanned by $\{r_2, r_3\}$. The subspace $\ker B \cap E^s(\beta_l)$ is one-dimensional and is therefore spanned by some vector

$$e = e_2 + e_3,\ e_m \in \text{span } r_m. \tag{5.20}$$

The vector space $BE^s(\beta_l)$ is one-dimensional and of real type; thus, we can write it as

$$BE^s(\beta_l) = \{X \in \mathbb{C}^2 : b \cdot X = 0\}, \tag{5.21}$$

where b is a suitable real nonzero row vector. We also choose a supplementary space of span e in $E^s(\beta_l)$:

$$E^s(\beta_l) = \check{E}(\beta_l) \oplus \text{span } e,\ \ \check{E}(\beta_l) = \text{span } \check{e},\ \text{ where } \check{e} = \check{e}_2 + \check{e}_3,\ \check{e}_m \in \text{span } r_m. \tag{5.22}$$

Thus, we have an isomorphism

$$B : \check{E}^s(\beta_l) \to BE^s(\beta_l). \tag{5.23}$$

For any k we write

$$\begin{aligned} E_{P_1}U_k &= \sigma_k^1(t,x,\theta_1)r_1 \\ E_{P_{in}}U_k &= \sigma_k^2(t,x,\theta_2)r_2 + \sigma_k^3(t,x,\theta_3)r_3, \end{aligned} \tag{5.24}$$

for scalar profiles σ_k^m. On the boundary we have using (5.22)

$$E_{P_{in}}U_k = \sigma_k^2(t,x_1,0,\theta_0)r_2 + \sigma_k^3(t,x_1,0,\theta_0)r_3 = a_k(t,x_1,\theta_0)e + \check{a}_k(t,x_1,\theta_0)\check{e} \tag{5.25}$$

for some scalar profiles a, $\check{a}$ with mean zero. We set

$$\check{U}_k = \check{a}_k\check{e}. \tag{5.26}$$

For a given H the boundary equation $BU_k = H(t,x_1,\theta_0)$ can now be rewritten

$$\begin{aligned} &a) B\underline{U_k} = \underline{H} \\ &b) B\check{U}_k = H^* - BE_{P_1}U_k - B[(I-E_P)U_k]^*. \end{aligned} \tag{5.27}$$

By (5.21) a solution $\check{U}_k$ to (5.27)(b) exists if and only if

$$b \cdot (H^* - BE_{P_1}U_k - B[(I-E_P)U_k]^*) = 0. \tag{5.28}$$

We write down now the equations that will be used to determine U_{-1}, U_0, and $E_{P_1}U_1$. The higher order equations follow the pattern that will be apparent. All profiles are required to be zero in $t < 0$.

First we have *interior* equations obtained by applying E_Q to equations in (5.18)

$$\begin{aligned} &a) E_Q[L(\partial)U_{-1} + e^{i\theta_3}MU_{-1}] = 0 \\ &b) E_Q[L(\partial)U_0 + e^{i\theta_3}MU_0] = 0 \\ &c) E_Q[L(\partial)U_1 + e^{i\theta_3}MU_1] = 0, \end{aligned} \tag{5.29}$$

and equations obtained by applying R to equations in (5.18)

$$
\begin{aligned}
&a)(I-E_P)U_{-1}=0\\
&b)(I-E_P)U_0=-R[L(\partial)U_{-1}+e^{i\theta_3}MU_{-1}]\\
&c)(I-E_P)U_1=-R[L(\partial)U_0+e^{i\theta_3}MU_0].
\end{aligned}
\tag{5.30}
$$

Each of the equations in (5.29) gives rise to four equations, one for each of E_{Q_i}, $i=0,1,2,3$.

We also have boundary equations

$$
B\underline{U_k}=0,\quad k=-1,0,1,
\tag{5.31}
$$

$$
\begin{aligned}
&a)b\cdot[-BE_{P_1}U_0+B[R(L(\partial)U_{-1}+e^{i\theta_3}MU_{-1})]^*]=0\\
&b)b\cdot[G-BE_{P_1}U_1+B[R(L(\partial)U_0+e^{i\theta_3}MU_0)]^*]=0\\
&c)b\cdot[-BE_{P_1}U_2+B[R(L(\partial)U_1+e^{i\theta_3}MU_1)]^*]=0
\end{aligned}
\tag{5.32}
$$

and

$$
\begin{aligned}
&a)B\check{U}_{-1}=-BE_{P_1}U_{-1}\\
&b)B\check{U}_0=-BE_{P_1}U_0+B[R(L(\partial)U_{-1}+e^{i\theta_3}MU_{-1})]^*\\
&c)B\check{U}_1=G-BE_{P_1}U_1+B[R(L(\partial)U_0+e^{i\theta_3}MU_0)]^*\\
&d)B\check{U}_2=-BE_{P_1}U_2+B[R(L(\partial)U_1+e^{i\theta_3}MU_1)]^*.
\end{aligned}
\tag{5.33}
$$

The equations (5.32), (5.33) are derived in the obvious way from (5.27), (5.28) using (5.30). The equations (5.32) are the respective solvability conditions for the equations (5.33)(b)-(d).

Remark 5.9. *The operator R acts on functions of $\theta=(\theta_1,\theta_2,\theta_3)$, so it must be understood that the restriction to $\theta_1=\theta_2=\theta_3=\theta_0$ in equations* (5.32), (5.33) *is done* after *the action of R.*

We will use the following proposition, which slightly modifies a classical result [Lax57], to simplify the terms $E_{Q_m}L(\partial)E_{P_m}U$ that arise in the analysis of the profile equations.

Proposition 5.10. *For $U\in\mathcal{H}^\infty$ let $E_{P_m}U=\sigma(t,x,\theta_m)r_m$ and let $X_{\phi_m}=\partial_{x_2}-\partial_\tau\omega_m(\beta_l)\partial_t-\partial_\eta\omega_m(\beta_l)\partial_{x_1}$ be the transport vector field associated to the phase ϕ_m.*[67]

We have

$$
E_{Q_m}L(\partial)E_{P_m}U=Q_m(L(\partial)\sigma(t,x,\theta_m)r_m)=(X_{\phi_m}\sigma)B_2r_m
\tag{5.34}
$$

Proof. For $\beta=(\tau,\eta)$ near β_l let $r_m(\beta)$ satisfy

$$
(\tau I+B_1\eta+B_2\omega_m(\beta))r_m(\beta)=0,\ r_m(\beta_l)=r_m.
\tag{5.35}
$$

Differentiating (5.35) with respect to τ and η, evaluating at β_l, and applying ℓ_m on the left we obtain:

$$
-\partial_\tau\omega_m(\beta_l)=\ell_mr_m\text{ and }-\partial_\eta\omega_m(\beta_l)=\ell_mB_1r_m.
\tag{5.36}
$$

Thus,

$$
\ell_m(\partial_t+B_1\partial_{x_1}+B_2\partial_{x_2})r_m=X_{\phi_m}.
\tag{5.37}
$$

□

[67]The vector field X_{ϕ_m} satisfies $X_{\phi_m}=(\partial_t+\underline{v}_m\cdot\partial_x)c_m$, where $c_m=(\partial_\xi\lambda_{k_m})^{-1}$ and $\underline{v}_m$ is the group velocity defined in (2.10). One can see this by differentiating $\tau+\lambda_{k_m}(\eta,\omega_m(\tau,\eta))=0$.

5.3 Table of modes

One of the main challenges in constructing approximate solutions that exhibit double amplification is the high degree of coupling among the equations (5.29)-(5.33). By making an assumption about the various nonzero modes that appear in the profiles, it will turn out that we are able to decouple the equations.

For each m, we now list the characteristic and noncharacteristic modes that we *expect* to find in U_m. The characteristic mode $-\theta_1$, for example, will appear if the Fourier series of U_m can possibly contain a term $c(t,x)e^{-i\theta_1}$ for some nonzero c. Similarly, the noncharacteristic mode $\theta_3 - \theta_2$ will appear if the Fourier series of U_m can possibly contain a term $c(t,x)e^{i(\theta_3-\theta_2)}$ for some nonzero c. If for a given U_m the mode $\theta_3 - \theta_2$ does not appear in the list, this means a term of the form $c(t,x)e^{i(\theta_3-\theta_2)}$ with $c \neq 0$ cannot possibly appear in the Fourier series of U_m.

The list is produced by first making a reasonable guess that takes into account the boundary data in (5.1), the single resonance (5.2), the profile equations, the nature of the exact solution, and the condition (5.3). For example, the expectation that no modes $k_i\theta_i$ with $k_i < -2$ should appear derives from the fact that in the exact solution to the singular system corresponding to the problem (5.1), all terms $V_k(x_2,\zeta)$ as in (3.31) with $k < -2$ are zero. In section 5.4 the list will be fully justified by the actual construction of profiles satisfying the profile equations whose nonzero Fourier modes lie in this list.

Characteristic modes:

$$\begin{aligned}
&a)U_{-1} : -\theta_2, -\theta_3\\
&b)U_0 : k\theta_2, k\theta_3, k \geq -2; -\theta_1\\
&c)U_1 : k\theta_2, k\theta_3, k \geq -2; -\theta_1\\
&d)U_2 : k\theta_2, k\theta_3, k \geq -2; -\theta_1,
\end{aligned} \tag{5.38}$$

and the pattern continues.

Noncharacteristic modes:[68]

$$\begin{aligned}
&a)U_{-1} : \text{none}\\
&b)U_0 : -\theta_2 + \theta_3\\
&c)U_1 : \text{add } -\theta_2 + 2\theta_3,\ k\theta_2 + \theta_3, k \geq 1, \text{and} -2\theta_2 + 2\theta_3\\
&d)U_2 : \text{add } -\theta_2 + 3\theta_3,\ k\theta_2 + 2\theta_3, k \geq 1, \text{and} -2\theta_2 + 3\theta_3,\\
&e)U_2 : \text{add } -\theta_2 + 4\theta_3,\ k\theta_2 + 3\theta_3, k \geq 1, \text{and} -2\theta_2 + 4\theta_3,
\end{aligned} \tag{5.39}$$

and the pattern continues. For example, the mode $-2\theta_2 + 2\theta_3$ for U_1 arises in the "interaction term" $-Re^{i\theta_3}MU_0$ of $(I - E_P)U_1$ as:[69]

$$\theta_3 - \theta_1 = \theta_3 + (-2\theta_2 + \theta_3) = -2\theta_2 + 2\theta_3. \tag{5.40}$$

[68]The term "add" used in (5.39) for U_m, $m \geq 1$ means that the noncharacteristic modes of U_m are obtained by taking the modes listed after U_m together with all the noncharacteristic modes of U_{m-1}. Thus, for example, the full set of noncharacteristic modes of U_2 consists of all the modes appearing in lines a-d of (5.39).

[69]The first equation in (5.40) represents the relation holding between the phases ϕ_m associated to the θ_m.

5.4 Construction of the profiles

We begin with a remark on notation and terminology.

Remark 5.11. *Recall that any function $U(t,x,\theta)\in\mathcal{H}^\infty$ can be written as in* (5.9) *as*

$$U(t,x,\theta)=\underline{U}(t,x)+\sum_{m=1}^{3}U^m(t,x,\theta_m)+U^{nc}(t,x,\theta_1,\theta_2,\theta_3). \tag{5.41}$$

Writing $U^m(t,x,\theta_m)=\sum_{p\in\mathbb{Z}}c_p^m(t,x)e^{ip\theta_m}$, we refer to $c_p^m e^{ip\theta_m}$ as the "p-mode" of U^m and to $\{c_p^m e^{ip\theta_m}, m=1,2,3\}$ as the "p-modes of U". We will apply this terminology to each profile U_k in the expansion (2.37) *of $u_a(t,x)$. In such a case we write*

$$U_k^m(t,x,\theta_m)=\sum_{p\in\mathbb{Z}}c_{k,p}^m(t,x)e^{ip\theta_m} \text{ and } U_{k,p}^m(t,x,\theta_m)=c_{k,p}^m(t,x)e^{ip\theta_m}. \tag{5.42}$$

When we speak, for example, of the "$-\theta_2+\theta_3$-mode of U_k", we are referring to the term $c(t,x)e^{i(-\theta_2+\theta_3)}$ in the Fourier series of U_k^{nc}.

1. Assumptions. In order to construct profiles satisfying the profile equations, we *assume* that profiles in $\mathcal{H}^\infty$ satisfying the profile equations to any order exist, and we *assume* that the nonzero modes of those profiles appear in the above table. As long as we can *construct* explicit profiles in $\mathcal{H}^\infty$ satisfying the profile equations which have only those nonzero modes, it won't matter at all what assumptions we made to construct them.[70] The process of construction will verify the non-obvious fact that our two assumptions are *consistent* with each other.

We are not concerned about the issue of uniqueness of profiles, because we have a procedure for showing that sufficiently high order approximate solutions are close in a precise sense to the exact solution (section 5.6), and we know the exact solution is unique. Recall that all profiles are required to be zero in $t<0$.

2. Determination of $(I-E_p)U_{-1}$ and $\underline{U_{-1}}$. By (5.30) we have $(I-E_P)U_{-1}=0$. Taking the mean of the equations in (5.29) and using (5.31), we obtain for any $k\geq -1$

$$\begin{aligned}&E_{Q_0}[L(\partial)U_k+e^{i\theta_3}MU_k]=0\\&B\underline{U_k}=0.\end{aligned} \tag{5.43}$$

Now $\underline{e^{i\theta_3}ME_PU_{-1}}=0$ by (5.3). Thus, for $k=-1$ the problem (5.43) reduces to $L(\partial)\underline{U_{-1}}=0$, $B\underline{U_{-1}}=0$, and hence $\underline{U_{-1}}=0$. [71]

3. Determination of $E_{P_1}U_{-1}$ and $\check{U}_{-1}$. From step **2** we conclude

$$U_{-1}=\sigma_{-1}^1(t,x,\theta_1)r_1+\sigma_{-1}^2(t,x,\theta_2)r_2+\sigma_{-1}^3(t,x,\theta_3)r_3, \tag{5.44}$$

for some scalar profiles σ_{-1}^m. Using assumption (5.38)(a) we see there is no resonance in the interaction term $e^{i\theta_3}MU_{-1}$, so we obtain from (5.29), Proposition 5.10, and the definition of E_{Q_1}

$$E_{Q_1}[L(\partial)U_{-1}+e^{i\theta_3}MU_{-1}]=(X_{\phi_1}\sigma_{-1}^1)B_2r_1=0. \tag{5.45}$$

[70] In particular, there is no problem of circularity.

[71] Without the condition $Mr_3=0$, the problem for $\underline{U_{-1}}$ would clearly be coupled to the problem for the -1-mode of $U_{-1}^3(t,x,\theta_3)$. Similar couplings would occur for profiles $\underline{U_j}$, $j\geq 0$.

Thus, $E_{P_1}U_{-1}=0$, since the vector field X_{ϕ_1} is outgoing and $\sigma^1_{-1}=0$ in $t<0$.

Similarly, we obtain

$$(5.46)\qquad \begin{aligned} &E_{Q_2}[L(\partial)U_{-1}+e^{i\theta_3}MU_{-1}]=X_{\phi_2}\sigma^2_{-1}B_2r_2=0\\ &E_{Q_3}[L(\partial)U_{-1}+e^{i\theta_3}MU_{-1}]=X_{\phi_3}\sigma^3_{-1}B_2r_3+e^{i\theta_3}\sigma^3_{-1}Q_3(Mr_3)=X_{\phi_3}\sigma^3_{-1}B_2r_3=0.\end{aligned}$$

It remains to determine the traces of σ^m_{-1}, $m=2,3$ on $x_2=0$. From (5.33)(a) we obtain $\check{U}_{-1}=0$. Summarizing, we have so far:

$$(5.47)\qquad U_{-1}=\sigma^2_{-1}(t,x,\theta_2)r_2+\sigma^3_{-1}(t,x,\theta_3)r_3,\ \ \check{U}_{-1}=0,\ \ \sigma^m_{-1}, m=2,3 \text{ undetermined.}$$

4. Determination of $E_{P_1}U_0$ and the -2-modes of U_0. The traces of $\sigma^m_{-1}, m=2,3$ are coupled to $E_{P_1}U_0$ by (5.32)(a). Writing

$$(5.48)\qquad \begin{aligned} &E_PU_0=\sigma^1_0(t,x,\theta_1)r_1+\sigma^2_0(t,x,\theta_2)r_2+\sigma^3_0(t,x,\theta_3)r_3,\ \text{where}\\ &\sigma^m_0(t,x,\theta_m)=\sum_{p=-2}^{\infty}\sigma^m_{0,p}(t,x)e^{ip\theta_m},\ m=1,2,3,\end{aligned}$$

we now determine the -2-modes of $E_{P_{in}}U_0$ and the -1-mode of $E_{P_1}U_0$, that is, $\sigma^2_{0,-2}e^{-i2\theta_2}$, $\sigma^3_{0,-2}e^{-i2\theta_3}$, and $\sigma^1_{0,-1}e^{-i\theta_1}$.

Using (5.20), (5.22), and (5.25) we obtain

$$(5.49)\qquad \begin{aligned} &\sigma^2_0(t,x_1,0,\theta_0)r_2=a_0(t,x_1,\theta_0)e_2+\check{a}_0(t,x_1,\theta_0)\check{e}_2\\ &\sigma^3_0(t,x_1,0,\theta_0)r_3=a_0(t,x_1,\theta_0)e_3+\check{a}_0(t,x_1,\theta_0)\check{e}_3.\end{aligned}$$

From (5.33)(b) we see that $\check{U}_0$ has no -2-mode; thus $\check{a}_{0,-2}(t,x_1)=0$.

Next we determine the -2-mode of $a_0(t,x_1,\theta_0)$, that is, $a_{0,-2}(t,x_1)e^{-i2\theta_0}$. The terms of (5.32)(b) involving E_{P_1} and M have no -2-modes. Thus, if we differentiate (5.32)(b) with respect to θ_0 and consider the -2-mode of the resulting equation, this reduces to the -2-mode of

$$(5.50)\qquad -b\cdot\partial_{\theta_0}G=b\cdot\partial_{\theta_0}BRL(\partial)U_0.$$

By (5.30)(b) the only -2-modes of U_0 are in E_PU_0 and thus in $E_{P_{in}}U_0$, so we obtain[72]

$$(5.51)\qquad (X_{Lop}a_{0,-2})e^{-i2\theta_0}=-b\cdot\partial_{\theta_0}G=-2ib\cdot g_{-2}(t,x_1)e^{-i2\theta_0},$$

which determines $a_{0,-2}$ (since $a_{0,-2}=0$ in $t<0$). Here

$$(5.52)\qquad X_{Lop}=c_0\partial_t+c_1\partial_{x_1},\ c_0\neq 0,\ \ c_j\in\mathbb{R}$$

is a characteristic vector field of the Lopatinski determinant. The derivation of (5.51) from (5.50) as well as the derivations (and solvability) of other equations involving X_{Lop} occurring below are discussed in section 5.5.

[72]The antiderivative in R is “undone” by ∂_{θ_0}.

We now know the traces of $\sigma^m_{0,-2}$, $m=2,3$. Using again the observation that the only -2-modes of U_0 are in $E_{P_{in}}U_0$, we can determine $\sigma^m_{0,-2}$, $m=2,3$ from the -2-modes of

$$\begin{aligned} &E_{Q_2}L(\partial)E_{P_2}U_0=0\\ &E_{Q_3}L(\partial)E_{P_3}U_0=0. \end{aligned} \tag{5.53}$$

That is, we determine $\sigma^m_{0,-2}$, $m=2,3$ by solving:

$$X_{\phi_m}\sigma^m_{0,-2}=0,\ \ (\sigma^m_{0,-2}|_{x_2=0})r_m=a_{0,-2}e_m. \tag{5.54}$$

By (5.30)(b) $(I-P_1)U^1_0=0$, so we may determine the -1-mode of $E_{P_1}U_0$ from the -1-mode of

$$E_{Q_1}[L(\partial)E_{P_1}U_0+e^{i\theta_3}MU_0]=0. \tag{5.55}$$

Taking account of the resonance, this gives

$$(X_{\phi_1}\sigma^1_{0,-1}\,e^{-i\theta_1}+c^1_0e^{-i\theta_1}\sigma^2_{0,-2})B_2r_1=0,\ \text{where}\ c^1_0=\ell_1Mr_2, \tag{5.56}$$

which yields $\sigma^1_{0,-1}$. Note that to produce a 1-mode in $E_{P_1}U_0$ we would need U_0 to have a $2\theta_2-2\theta_3$ mode, but (5.30)(b) shows it does not. Similarly, $E_{P_1}U_0$ has no k-modes for $k>1$; thus, we now have $E_{P_1}U_0$.

5. Determination of U_{-1}, $(I-E_P)U_0$, $\underline{U_0}$, and $\check{U}_0$. Parallel to (5.49) we have

$$\sigma^m_{-1}(t,x_1,0,\theta_0)r_m=a_{-1}(t,x_1,\theta_0)e_m,\ m=2,3. \tag{5.57}$$

Using (5.32)(a), we determine the -1-mode of a_{-1}, namely $a_{-1,-1}e^{-i\theta_0}$, from the -1-mode of[73]

$$\partial_{\theta_0}(b\cdot BRL(\partial)U_{-1})=\partial_{\theta_0}(b\cdot BE_{P_1}U_0), \tag{5.58}$$

that is,

$$(X_{Lop}a_{-1,-1})e^{-i\theta_0}=\partial_{\theta_0}(b\cdot Be^{-i\theta_0}\sigma^1_{0,-1}r_1). \tag{5.59}$$

In view of (5.57) we now know $\sigma^m_{-1,-1}(t,x_1,0)e^{-i\theta_m}$, $m=2,3$, so we can determine $\sigma^m_{-1,-1}(t,x)e^{-i\theta_m}$, $m=2,3$, using the -1-modes of the equations (5.46).

Knowing $E_{P_1}U_0$, we proceed to determine the higher ($k\geq 1$) modes of $a_{-1}(t,x_1,\theta_0)$ using "∂_{θ_0}(5.32)(a)":

$$X_{Lop}a_{-1}=\partial_{\theta_0}[b\cdot(BE_{P_1}U_0-BRe^{i\theta_3}MU_{-1})]. \tag{5.60}$$

The 1-mode of the right side of (5.60) is zero, so $a_{-1,1}=0$. Similarly, the higher modes of a_{-1} are also zero. Having a_{-1} we can complete the determination of U_{-1} using the interior equations (5.46). We note that the nonzero modes of U_{-1} lie in the list of section 5.3.

Next we use (5.30)(b) to determine $(I-E_P)U_0$ from U_{-1}. From this we see that $(I-E_P)U_0$ has characteristic (respectively, noncharacteristic) modes only of the forms

$$-\theta_3,-\theta_2,\ \text{and}\ -\theta_2+\theta_3 \tag{5.61}$$

[73]The right side of (5.58) has no -2-mode, so $a_{-1,-2}=0$.

respectively. Knowing $E_{P_1}U_0$ and U_{-1}, we determine $\check{U}_0$ from (5.33)(b). Knowing $(I-E_P)U_0$, we can determine the possibly nonzero mean $\underline{U_0}$ from the problem (5.43) in the case $k=0$.

To complete the determination of $\check{U}_0$ we need the higher ($k \geq -1$) modes of $E_{P_{in}}U_0$.

6. Determination of $E_{P_1}U_1$, the -2-modes of U_1, and the -1-modes of $E_{P_{in}}U_0$. Using (5.30)(c) we determine the -2-modes of $(I-E_P)U_1$, that is, the -2-modes of $(I-P_m)U_1^m(t,x,\theta_m)$, $m=2,3$.

Parallel to (5.49) we have

$$\sigma_1^m(t,x_1,0,\theta_0)r_m = a_1(t,x_1,\theta_0)e_m + \check{a}_1(t,x_1,\theta_0)\check{e}_m,\ m=2,3. \tag{5.62}$$

From (5.33)(c) we can determine the -2-mode of $\check{a}_1$, that is $\check{a}_{1,-2}(t,x_1)e^{-i2\theta_0}$. We determine the -2-mode of a_1, that is $a_{1,-2}(t,x_1)e^{-i2\theta_0}$, from the -2-mode of "∂_{θ_0}(5.32)(c)". This equation reduces to the -2-mode of

$$\partial_{\theta_0}[b\cdot BRL(\partial)(E_{P_{in}}U_1 + (I-E_P)U_1)] = 0, \tag{5.63}$$

where the -2-mode of the $(I-E_P)U_1$ term is known.

Having the traces of $\sigma_{1,-2}^m$, $m=2,3$, we now use the -2-modes of (5.29)(c), which reduce to the -2-modes of

$$E_{Q_{in}}[L(\partial)(E_{P_{in}}U_1 + (I-E_P)U_1)] = 0, \tag{5.64}$$

to finish the determination of $\sigma_{1,-2}^m$, $m=2,3$. We now have the -2-modes of U_1.

Letting $U_{1,-2}^2(t,x,\theta_2)$ denote the -2-mode of $U_1^2(t,x,\theta_2)$ and taking account of the resonance, we can now determine the -1-mode of $E_{P_1}U_1$, that is, $\sigma_{1,-1}^1(t,x)e^{-i\theta_1}$ from the -1-mode of

$$E_{Q_1}[L(\partial)(E_{P_1}U_1 + (I-P_1)U_1^1) + e^{i\theta_3}MU_{1,-2}^2] = 0. \tag{5.65}$$

Note that the -1-mode of $(I-P_1)U_1^1$ is known from (5.30)(c) and the result of step **4**.

To complete the determination of $E_{P_1}U_1$ we use the equations

$$\begin{aligned}&(a)(I-E_P)U_1 = -R[L(\partial)U_0 + e^{i\theta_3}MU_0]\\&(b)E_{Q_1}[L(\partial)(E_{P_1}U_1 + (I-P_1)U_1^1) + e^{i\theta_3}MU_1] = 0.\end{aligned} \tag{5.66}$$

By (5.66)(b) in order produce a 1-mode in $E_{P_1}U_1$ we would need either U_1 to have a $2\theta_2-2\theta_3$-mode or $(I-P_1)U_1^1$ to have a 1-mode. By (5.66) (a) neither of these possibilities occurs. Similarly, $E_{P_1}U_1$ has no k-modes for $k>1$; thus, we now have $E_{P_1}U_1$.

Having $E_{P_1}U_1$ we proceed to determine the -1-modes of $E_{P_{in}}U_0$, starting with their traces (recall (5.49)). For this we consider the -1-mode of "∂_{θ_0}(5.32)(b)", which reduces to the -1-mode of

$$\partial_{\theta_0}\{b\cdot[-BE_{P_1}U_1 + B[R(L(\partial)U_0 + e^{i\theta_3}M(U_{0,1}^2 + U_{0,1}^3))]]\} = 0. \tag{5.67}$$

We can write

$$U_0 = E_{P_{in}}U_0 + \underline{U_0} + E_{P_1}U_0 + (I-E_P)U_0, \tag{5.68}$$

where the last three terms on the right are known. Since $\check{U}_0$ is known, the -1-mode of equation (5.67) reduces to a propagation equation involving X_{Lop} for the only unknown, the -1-mode of $a_0(t,x_1,\theta_0)$.

We now have the traces of $\sigma^m_{0,-1}e^{-i\theta_m}$, $m=2,3$. To determine these modes we consider the -1-modes of (5.29)(b). The relevant equations reduce to the -1-modes of the incoming equations

$$(5.69)\qquad \begin{aligned}&E_{Q_2}[L(\partial)(E_{P_2}U_0+(I-E_P)U_0)]=0\\ &E_{Q_3}[L(\partial)(E_{P_3}U_0+(I-E_P)U_0)+e^{i\theta_3}MU^3_{0,-2}]=0,\end{aligned}$$

where the only unknowns are the -1-modes of $E_{P_m}U_0$, $m=2,3$. These are now determined.

7. Determination of U_0, $(I-E_P)U_1$, $\underline{U_1}$, and $\check{U}_1$. We now complete the determination of U_0, by first determining the higher modes of $a_0(t,x_1,\theta_0)$ (as in (5.49)). For this we use the boundary equation

$$(5.70)\qquad \partial_{\theta_0}\{b\cdot[G-BE_{P_1}U_1+BR(L(\partial)U_0+e^{i\theta_3}MU_0)]\}=0.$$

Since the only unknown piece of U_0 is $E_{P_{in}}U_0$, the only unknown in (5.70) is a_0. By the analysis of section 5.5, this equation turns out to have the form

$$(5.71)\qquad X_{Lop}a_0+e^{i\theta_0}m(D_{\theta_0})a_0=g(t,x_1,\theta_0),$$

where g is known and $m(D_{\theta_0})$ is a *bounded* Fourier multiplier. The equation has a solution and is expected to have nonzero k-modes for $k\geq 1$ in addition to the already known -2 and -1-modes.[74]

In view of (5.49) we now have the trace of $E_{P_{in}}U_0$, so we can determine $E_{P_{in}}U_0$ by solving the incoming equations

$$(5.72)\qquad \begin{aligned}&E_{Q_2}[L(\partial)(E_{P_2}U_0+(I-E_P)U_0)]=0\\ &E_{Q_3}[L(\partial)(E_{P_3}U_0+(I-E_P)U_0)+e^{i\theta_3}M(E_{P_3}U_0+(I-E_P)U_0)]=0.\end{aligned}$$

This gives U_0. We determine $(I-E_P)U_1$ and $\check{U}_1$ from (5.30)(c) and (5.33)(c), respectively. Having $(I-E_P)U_1$ we determine the possibly nonzero mean $\underline{U_1}$ using the system (5.43) in the case $k=1$.

Observe that the nonzero modes of U_0 lie in the list of section 5.3. Although U_1 is not yet completely determined, (5.30)(c) implies that the nonzero noncharacteristic modes of U_1 must lie in that list.

8. Determination of $E_{P_1}U_2$, the -2-modes of U_2, and the -1-modes of $E_{P_{in}}U_1$. Except for obvious small changes, the determination of these modes is by an almost verbatim repetition of step **6.** For example, instead of considering the -2-mode of (5.63), one now considers the -2-mode of

$$(5.73)\qquad \partial_{\theta_0}[b\cdot BRL(\partial)(E_{P_{in}}U_2+(I-E_P)U_2)]=0.$$

9. Determination of U_1, $(I-E_P)U_2$, $\underline{U_2}$, and $\check{U}_2$. This step is a near repetition of step **7.** In place of (5.66) we now use

$$(5.74)\qquad \begin{aligned}&(a)(I-E_P)U_2=-R[L(\partial)U_1+e^{i\theta_3}MU_1]\\ &(b)E_{Q_1}[L(\partial)(E_{P_1}U_2+(I-P_1)U_2^1)+e^{i\theta_3}MU_2]=0.\end{aligned}$$

10. Conclusion. The inductive pattern is now clear. For any M this argument constructs profiles U_{-1}, U_0, ..., U_M in $\mathcal{H}^\infty$ satisfying the profile equations (5.18), (5.19). Moreover, it is evident from the construction that the nonzero modes of these profiles lie in the list of section 5.3.

[74]Since $\underline{U_0}$ may be nonzero, the interaction term in (5.70) contributes a 1-mode, so a_0 (and hence U_0) may have a nonzero 1-mode. But then the interaction term may contribute a nonzero 2-mode, etc...

5.5 Analysis of the boundary amplitude equations

In this section we show that for each $j \geq -1$ the boundary equation for the amplitude $a_j(t, x_1, \theta_0)$ in the construction of section 5.4 is an equation of the form

$$X_{Lop} a_j + e^{i\theta_0} m_j(D_{\theta_0}) a_j = g_j(t, x_1, \theta_0), \quad g_j = 0 \text{ in } t < 0, \tag{5.75}$$

where $g_j(t, x_1, \theta_0) = \sum_{k \geq -2} g_{j,k}(t, x_1) e^{ik\theta_0}$ is known, $m_j(D_{\theta_0})$ is a *bounded* Fourier multiplier, and [75]

$$X_{Lop} = c_0 \partial_t + c_1 \partial_{x_1} \text{ with } c_j \in \mathbb{R} \text{ and } c_0 \neq 0. \tag{5.76}$$

The equation for a_j derives from an equation of the form[76]

$$\partial_{\theta_0} \{ b \cdot B[R(L(\partial) U_j + e^{i\theta_3} M U_j)] \} = h_j(t, x_1, \theta_0), \tag{5.77}$$

where h_j is known from previous steps, and $U_j \in \mathcal{H}^\infty$ can be written:

$$U_j = \underline{U_j} + E_{P_{in}} U_j + E_{P_1} U_j + (I - E_P) U_j. \tag{5.78}$$

The terms $\underline{U_j} + E_{P_1} U_j + (I - E_P) U_j$ are also known from previous steps, and on $x_2 = 0$ we may write (recall (5.25))

$$\begin{aligned} E_{P_{in}} U_j(t, x_1, 0, \theta) &= \sigma_j^2(t, x_1, 0, \theta_2) r_2 + \sigma_j^3(t, x_1, 0, \theta_3) r_3 = \\ & (a_j(t, x_1, \theta_2) e_2 + \check{a}_j(t, x_1, \theta_2) \check{e}_2) + (a_j(t, x_1, \theta_3) e_3 + \check{a}_j(t, x_1, \theta_3) \check{e}_3), \end{aligned} \tag{5.79}$$

where $\check{a}_j$, too, is known from previous steps. Dropping all subscripts j and using that fact that $R_m B_2 P_m = 0$ for $m = 2, 3$, we see then that (5.77) reduces to an equation of the form

$$\begin{aligned} \partial_{\theta_0} \left[b \cdot BR \left(L'(\partial)(a(t, x_1, \theta_2) e_2 + a(t, x_1, \theta_3) e_3) + e^{i\theta_3} M(a(t, x_1, \theta_2) e_2 + a(t, x_1, \theta_3) e_3) \right) \right] = \\ h(t, x_1, \theta_0), \quad h = 0 \text{ in } t \leq 0, \end{aligned} \tag{5.80}$$

where h is known and $L'(\partial) = \partial_t + B_1 \partial_{x_1}$.

It is shown in [CG10] that

$$\partial_{\theta_0} [b \cdot BRL'(\partial)(a(t, x_1, \theta_2) e_2 + a(t, x_1, \theta_3) e_3)] = X_{Lop} a(t, x_1, \theta_0) \tag{5.81}$$

for X_{Lop} as in (5.76). Clearly,[77]

$$\partial_{\theta_0} [b \cdot BR(e^{i\theta_3} M a(t, x_1, \theta_3) e_3)] = \alpha_1 e^{i\theta_0} a(t, x_1, \theta_0) \text{ for } \alpha_1 = b \cdot BR_3 M e_3 = 0, \tag{5.82}$$

so it remains to analyze the term $\partial_{\theta_0} [b \cdot BR \left(e^{i\theta_3} M a(t, x_1, \theta_2) e_2 \right)]$.

Writing $a(t, x_1, \theta_0) = \sum_{k \geq -2} a_k(t, x_1) e^{ik\theta_0}$, we have (with $\beta_1 = b \cdot BR_1 M e_2$)[78]

$$\begin{aligned} \partial_{\theta_0} [b \cdot BR \left(e^{i\theta_3} M a(t, x_1, \theta_2) e_2 \right)] &= \partial_{\theta_0} [b \cdot BR(M e_2 \sum_{k \geq -2} a_k e^{ik\theta_2 + i\theta_3})] = \\ \beta_1 e^{-i\theta_0} a_{-2}(t, x_1) &+ \partial_{\theta_0} [b \cdot B \sum_{k \geq 1} (L^{-1}(ikd\phi_2 + id\phi_3) M e_2) a_k e^{i(k+1)\theta_0}] = \\ \beta_1 e^{i\theta_0} (a_{-2}(t, x_1) e^{-i2\theta_0}) &+ b \cdot B \sum_{k \geq 1} (L^{-1}(kd\phi_2 + d\phi_3) M e_2)(k+1) a_k e^{i(k+1)\theta_0}. \end{aligned} \tag{5.83}$$

We can simplify the second term in the last line using

[75] In fact, X_{Lop} is a characteristic vector field of the Lopatinski determinant [CG10].
[76] As usual, the bracketed term on the left in (5.77) is restricted to $\theta_2 = \theta_3 = \theta_0$ after the action of R.
[77] Recall $M r_3 = 0$.
[78] The terms with $k = -1, 0$ in the second line of (5.83) are zero; recall $a_0 = 0$.

Lemma 5.12. *For each $k \in \{1, 2, \dots\}$ the number $k(\omega_2 - \omega_1) + (\omega_3 - \omega_1)$ is nonzero, and for any $X \in \mathbb{C}^3$*

$$L^{-1}(kd\phi_2 + d\phi_3)X = c_1 r_1 + c_2 r_2 + c_3 r_3, \text{ where } c_1 = \frac{\ell_1 X}{k(\omega_2 - \omega_1) + (\omega_3 - \omega_1)}. \tag{5.84}$$

Proof. We can write $L(d\phi_m) = \sum_{m \neq m'} (\omega_m - \omega_{m'}) B_2 P_{m'}$, so $L(d\phi_m) r_p = (\omega_m - \omega_p) B_2 r_p$ and

$$L(kd\phi_2 + d\phi_3) r_1 = (k(\omega_2 - \omega_1) + (\omega_3 - \omega_1)) B_2 r_1. \tag{5.85}$$

For $k \in \{1, 2, \dots\}$ it follows that $k(\omega_2 - \omega_1) + (\omega_3 - \omega_1) \neq 0$, since otherwise $L(kd\phi_2 + d\phi_3)$ would have a nontrivial kernel.

Thus, (5.84) follows by computing

$$\ell_1 X = \ell_1 L(kd\phi_2 + d\phi_3)(c_1 r_1 + c_2 r_2 + c_3 r_3) = c_1[k(\omega_2 - \omega_1) + (\omega_3 - \omega_1)]. \tag{5.86}$$

□

Using the lemma and the fact that $b \cdot Br_p = 0$, $p = 2, 3$, we obtain

$$\begin{aligned} &b \cdot B \sum_{k \geq 1} (L^{-1}(kd\phi_2 + d\phi_3) M e_2)(k+1) a_k e^{i(k+1)\theta_0} = \\ &(b \cdot Br_1)(\ell_1 M e_2) e^{i\theta_0} \sum_{k \geq 1} \frac{(k+1)}{k(\omega_2 - \omega_1) + (\omega_3 - \omega_1)} a_k e^{ik\theta_0}. \end{aligned} \tag{5.87}$$

The set of numbers $\{\frac{(k+1)}{k(\omega_2 - \omega_1) + (\omega_3 - \omega_1)}, k \in \{1, 2, \dots\}\}$ is clearly bounded. Thus, the equation (5.80) takes the form

$$X_{Lop} a + e^{i\theta_0} m(D_{\theta_0}) a = h(t, x_1, \theta_0),\ h = 0 \text{ in } t < 0, \tag{5.88}$$

where the components $m(k)$ defining the bounded Fourier multiplier $m(D_{\theta_0})$ can be read off from (5.83) and (5.87). A standard argument based on a simple energy estimate yields a unique solution $a(t, x_1, \theta_0) \in H^\infty((-\infty, T] \times \mathbb{R} \times \mathbb{T})$ satisfying $a = 0$ in $t < 0$.

Remark 5.13. *In section 5.4 the modes $a_{j,-2} e^{-i2\theta_0}$, $a_{j,-1} e^{-i\theta_0}$ were determined for each a_j by simple equations like* (5.51), (5.59) before *the full profile $a_j(t, x_1, \theta_0)$ was determined by an equation like* (5.75).[79] *It is easy to check that the -2 and -1-modes of the solution to* (5.75) *agree with the previously determined modes.*

5.6 Justification of the approximate solution of Example 2.14.

We now complete the proof of parts (a),(b),(c) of Example 2.14.

1. Part (a). The existence and uniqueness of an exact solution $u^\epsilon(t, x) \in H^\infty(\Omega_T)$ to (2.35) (or (5.1)) follows by applying the results of [Cou05] (for WR problems without highly oscillatory coefficients) to the problem obtained from (2.35) by *fixing* any particular $\epsilon \in (0, \epsilon_0]$.

[79] We found $a_{-1,-2} = 0$ in step **5** of the construction of section 5.4.

Theorem 2.11(b) yields an exact solution $U^\epsilon(t,x,\theta_0)\in L^2$ to the singular problem (1.7) corresponding to (2.35).[80] One can prove higher regularity of U^ϵ (in fact, $U^\epsilon\in H^\infty(\Omega_T\times\mathbb{T})$) by differentiating the singular problem and repeating the argument of section 6 of [Wil20]. Since $v^\epsilon := U^\epsilon(t,x,\theta_0)|_{\theta_0=\frac{\phi_0}{\epsilon}}$ is a solution of (2.35) on $(-\infty,T]$, we conclude $u^\epsilon = v^\epsilon$.

2. Part (b). The construction carried out in sections 5.1-5.5 yields profiles $U_k(t,x,\theta_1,\theta_2,\theta_3)$ as in (2.37) satisfying the profile equations (5.18), (5.19) for $j=1,\dots,J$, where J is as large as desired. With $\theta=(\theta_1,\theta_2,\theta_3)$ define

$$
\begin{aligned}
&\mathcal{U}(t,x,\theta)=\sum_{k=-1}^{J}\epsilon^k U_k(t,x,\theta),\\
&U_a^\epsilon(t,x,\theta_0)=\mathcal{U}\left(t,x,\theta_0+\frac{\omega_1x_2}{\epsilon},\theta_0+\frac{\omega_2x_2}{\epsilon},\theta_0+\frac{\omega_3x_2}{\epsilon}\right),
\end{aligned}\tag{5.89}
$$

and observe that for any given $M>0$ one can choose $J=J(M)$ so that U_a^ϵ satisfies

$$
\begin{aligned}
&D_{x_2}U_a^\epsilon + A_0(D_t+\frac{\sigma_l}{\epsilon}D_{\theta_0})U_a^\epsilon + A_1(D_{x_1}+\frac{\eta_l}{\epsilon}D_{\theta_0})U_a^\epsilon - ie^{i\left(\frac{\omega_3(\beta_l)}{\epsilon}x_2+\theta_0\right)}B_2^{-1}MU_a^\epsilon=\epsilon^M R_M^\epsilon(t,x,\theta_0)\\
&BU_a^\epsilon=\epsilon g_{-2}(t,x_1)e^{-i2\theta_0}+\epsilon^M r_M^\epsilon(t,x_1,\theta_0)\text{ on }x_2=0\\
&U_a=0\text{ in }t<0.
\end{aligned}\tag{5.90}
$$

Here (with obvious notation) the error terms satisfy for any α:

$$
|(\epsilon\partial_{x_2},\partial_{t,x_1,\theta_0})^\alpha R_M|_{L^2_T(t,x,\theta_0)}\le C_\alpha,\quad |(\partial_{t,x_1,\theta_0})^\alpha r_M|_{L^2_T(t,x_1,\theta_0)}\le C_\alpha.\tag{5.91}
$$

Next introduce the higher norm:

$$
|V|^2_{0,m}:=\int_0^\infty |V(t,x_1,x_2,\theta_0)|^2_{H^m(t,x_1,\theta_0)}dx_2\text{ for }m\in\mathbb{N}.\tag{5.92}
$$

We showed in section 6 of [Wil20] that the estimate (2.29) of Theorem 2.11 can be upgraded to a higher derivative estimate: there exist constants K, γ_0 independent of ϵ such that for $\gamma\ge\gamma_0$ and U as in Theorem 2.11:

$$
|U^\gamma|_{0,m}+|\partial_{x_2}U^\gamma|_{0,m}\le\frac{K}{(\epsilon\gamma)^{\mathbb{E}}}\left[\frac{1}{\epsilon\gamma^2}|\Lambda_D F^\gamma|_{0,m+1}+\frac{1}{\epsilon\gamma^{3/2}}\langle\Lambda_D G^\gamma\rangle_{m+1}\right],\tag{5.93}
$$

where $\mathbb{E}$ is now computed from (2.30) to be $\mathbb{E}=1$.[81] For $m>\frac{3}{2}$ the left side of (5.93) dominates $|U^\gamma|_{L^\infty(\Omega_T\times\mathbb{T})}$, so the estimate (5.93) can be applied directly to the error problem satisfied by $E^\epsilon=U^\epsilon-U_a^\epsilon$ to show that E^ϵ is $O(\epsilon^Q)$ in $L^\infty(\Omega_T\times\mathbb{T})$, provided M in (5.90) is large enough (see Remark 5.14, (3)). This implies the estimate (2.38).

3. Part (c). This part follows from an examination of steps **3-5** of the profile construction in section 5.4. Equation (5.51) implies $a_{0,-2}\neq 0$ in $t>0$ as long as $b\cdot g_{-2}\neq 0$ in $t>0$.[82] Then (5.54) implies $\sigma^2_{0,-2}\neq 0$ in $t>0$. Next (5.56) implies $\sigma^1_{0,-1}\neq 0$ in $t>0$ as long as $c_0^1=\ell_1Mr_2\neq 0$. Then (5.59) implies $a_{-1,-1}\neq 0$ in $t>0$, and finally (5.46) implies $\sigma^m_{-1,-1}\neq 0$ in $t>0$ for $m=2,3$.

[80] To apply Theorem 2.11(b) to the system (2.35), we must take an extension of $g_{-2}(t,x_1)$ to $t>T$. This standard maneuver is used also in step **2** below; for more detail, see p. 586 of [CGW14], for example.

[81] In (2.30) we now have $P=1$, $|\mathcal{I}|=2$, $|\mathcal{O}|=1$, $|\Upsilon_0|=2$.

[82] When we say here that "$a_{0,-2}\neq 0$ in $t>0$", we mean that $a_{0,-2}$ takes nonzero values for arbitrarily small $t>0$; the same applies to other functions.

Remark 5.14. ***1. Optimality of estimates.*** *Estimates for which* $\mathbb{E} = 0$ *(for example, when* $|\mathcal{I}| = 1$, $|\Upsilon^0| = 2$ *as in Theorem 2.12) are clearly optimal. Simple examples show that amplification due to the factors* $|X_k|$ *is unavoidable in WR problems [CG10] even when the oscillatory coefficient* $\mathcal{D}$ *is zero.*

The optimality of the estimate (2.28) *for the singular problem corresponding to* (2.35) *(with* $\mathbb{E} = 1$ *given by* (2.30)*) is confirmed by the observation that* U_a^ϵ *as in* (5.89) *satisfies*

$$|U_a^\epsilon(t, x, \theta_0)|_{L^2(\Omega_T \times \mathbb{T})} = O\left(\frac{1}{\epsilon}\right), \tag{5.94}$$

and so U_a^ϵ *is amplified by the factor* $\frac{1}{\epsilon^2}$ *relative to the* $O(\epsilon)$ *boundary data of* (2.35)*. In* (2.28) *one factor of* $\frac{1}{\epsilon}$ *is contributed by the factor* $|X_{-2}|$ *on* $\widehat{G_{-2}} = \widehat{\epsilon g_{-2}}$*, and a second factor is contributed by* $\frac{1}{(\epsilon\gamma)^{\mathbb{E}}}$*, so the estimate "predicts" exactly the order of amplification exhibited by* U_a^ϵ*, and hence also by the exact solution.*

2. Triple and higher amplification*. Step* ***3*** *of the proof in section 5.6 verifies double amplification, instantaneous production of a (nonzero) incoming* $\frac{1}{\epsilon}\sigma^2_{-1,-1}(t, x)e^{-i\theta_2}$ *mode in the leading term,* U_{-1}*, of the approximate solution. In order to achieve triple amplification one could modify the oscillatory coefficient in* (5.1) *to be* $(e^{i\frac{\phi_3}{\epsilon}} + e^{i\frac{2\phi_3}{\epsilon}})M$ *and take* $\epsilon g_{-4}(t, x_1)e^{-i4\frac{\phi_0}{\epsilon}}$ *as the boundary datum. The resonance* $-4\phi_2 + 2\phi_3 = -2\phi_1$ *would then be expected, by arguments like those in the above construction, to produce an outgoing* $\sigma^1_{0,-2}e^{-i2\theta_1}$ *mode in* U_0*, and then a "reflected" incoming* $\frac{1}{\epsilon}\sigma^2_{-1,-2}e^{-i2\theta_2}$ *mode in* U_{-1}*. The resonance* $-2\phi_2 + \phi_3 = -\phi_1$ *should then produce an outgoing* $\frac{1}{\epsilon}\sigma^1_{-1,-1}(t, x)e^{-i\theta_1}$ *mode in* U_{-1} *that reflects to produce an incoming* $\frac{1}{\epsilon^2}\sigma^2_{-2,-1}(t, x)e^{-i\theta_2}$ *mode in the leading term* U_{-2}*.*

Similarly, modifying the oscillatory coefficient in (5.1) *to be* $(e^{i\frac{\phi_3}{\epsilon}} + e^{i\frac{2\phi_3}{\epsilon}} + e^{i\frac{4\phi_3}{\epsilon}})M$ *and taking* $\epsilon g_{-8}(t, x_1)e^{-i8\frac{\phi_0}{\epsilon}}$ *as the boundary datum should result in the production of a* $\frac{1}{\epsilon^3}\sigma^2_{-3,-1}(t, x)e^{-i\theta_2}$ *mode in the leading term* U_{-3} *(instantaneous fourth-order amplification); and so on. With each successive modification of this sort in the oscillatory coefficient, the formula* (2.30) *shows that* $\mathbb{E}$ *increases by one. Thus, these higher interactions would exhibit the optimality of* (2.28) *with* $\mathbb{E}$ *given by* (2.30) *for larger values of* P*, at least for problems like* (5.1)*.*

Finally, note that since $-2\omega_2 + \omega_3 = -\omega_1$*, we have* $\Omega_{1,2} = \frac{\omega_1 - \omega_3}{\omega_2 - \omega_1} = -2$*, so the first possibility in the hypothesis* (4.41) *of Proposition 4.6 holds here. That proposition, taken together with this discussion of multiple amplification, suggests that when* $\Omega_{i,j}$ *is rational and lies in* $(0, \infty)$ *or* $(-\infty, -1)$*, there is no hope of proving an estimate like* (2.28) *with finite* $\mathbb{E}$ *for problems where the spectrum of* $d(\theta_{in})$ *is an arbitrary infinite subset of* $\mathbb{N}$*. One should expect multiple amplification of infinite order in the presence of such resonances for certain choices of* $d(\theta_{in})$ *with infinite spectrum in* $\mathbb{N}$*.*

3. γ_0 ***independent of*** ϵ***.*** *In step* ***2*** *of the proof of section 5.6, to conclude* $|E^\epsilon|_{L^\infty(\Omega_T \times \mathbb{T})} = O(\epsilon^Q)$ *we apply* (5.93) *with* $\gamma = \gamma_0$ *to the problem satisfied by* E^ϵ*, use*

$$e^{-\gamma_0 T}|E^\epsilon|_{L^\infty(\Omega_T \times \mathbb{T})} \leq |e^{-\gamma_0 t}E^\epsilon|_{0,m} + |e^{-\gamma_0 t}\partial_{x_2}E^\epsilon|_{0,m} \tag{5.95}$$

and then multiply both sides of the resulting estimate by $e^{\gamma_0 T}$*. Here we see it is crucial that* $\gamma_0 \sim 1$*. If one tried to use the method of [CGW14] (simultaneous diagonalization) to estimate* E^ϵ*, one would have to take* $\gamma_0 \sim \frac{1}{\epsilon}$*, and the factor* $e^{\gamma_0 T}$ *would then overwhelm the terms* $\epsilon^M R^\epsilon_M$ *and* $\epsilon^M r^\epsilon_M$ *coming from the right side of* (5.93)*. A similar problem arises if one tries to use the estimates of [Cou05] to estimate* $|u^\epsilon - u^\epsilon_a|_{L^\infty(\Omega_T)}$ *for each fixed* ϵ*.*

6 Discussion

Let us assess what our results suggest about the prospects of proving uniform estimates (Remark 1.1) for problems like (1.3) obtained as linearizations of quasilinear problems. This paper has dealt only with wavetrain (as opposed to pulse) solutions, so we restrict our comments to such solutions.

Whenever the oscillatory function v in the coefficients of (1.3) is real-valued, one cannot avoid two-sided cascades, since such functions have both positive and negative Fourier spectrum. Our estimate for problem (1.1) in the two-sided case, Theorem 2.12, requires that $\Upsilon_0 = \{\pm\beta_l\}$ and that there is only one incoming phase, $\mathcal{I} = \{N\}$. This result offers hope of proving a similar estimate for (linearizations of) quasilinear problems that satisfy those conditions, as well as for problems like the vortex sheet problem for which there is only one incoming phase.

Our example of double amplification, Example 2.14, assumed the existence of a resonance for which

$$\Omega_{i,j} \in ((-\infty, -1) \cup (0, \infty)) \cap \mathbb{Q}; \tag{6.1}$$

in that example $\Omega_{1,2} = -2$. It is clear that resonances are generically *absent* [CW17]. In the absence of resonances our results leave open the possibility of proving estimates like that of Theorem 2.12 for more general problems with two-sided cascades, including problems where $|\Upsilon_0| > 2$ or $|\mathcal{I}| > 1$.

There are at least two reasons to study one-sided cascades. First, this is the simplest context in which to observe the interesting phenomenon of multiple amplification. A second, related, reason is that the occurrence of multiple amplification allows us to confirm that there are situations in which large global amplification factors $\mathbb{D}(\epsilon, k, k-r)(\zeta)$ (equal to $\frac{C_5|r|^3}{\epsilon\gamma}$ on $\zeta-$sets of positive measure) are *activated*. The confirmation is provided by having explicit, multiply amplified, approximate solutions that we know are close in the sense of Example 2.14 to exact solutions $u^\epsilon(t,x) = U^\epsilon(t,x,\frac{\phi_0}{\epsilon})$. Since U^ϵ satisfies an estimate of the form (2.28), we conclude that the exponent $\mathbb{E}$ in that estimate must be ≥ 1, and thus some factors $\mathbb{D}(\epsilon, k, k-r)$ must be large. So far we have demonstrated the activation of large amplification factors only when there is a resonance satisfying (6.1).

This information has a direct bearing on problems with two-sided cascades, since the *same* amplification factors occur in those problems. Indeed, the results of section 3 show that the factors $\mathbb{D}(\epsilon, k, k-r)$ are determined just by our assumptions on $(L(\partial), B)$; they are independent of the choice of the oscillatory factor $\mathcal{D}(\theta_{in})$. The discussion in section 1.2.1 and in part 2 of Remark 5.14 indicates how the activation of large amplification factors in a problem with two-sided cascades may rule out any estimate of the form (2.28) with finite $\mathbb{E}$. There is a clear, but as yet unverified, mechanism for multiple amplification of infinite order.

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