STABILITY OF MULTIDIMENSIONAL VISCOUS SHOCKS C.I.M.E. LECTURES, CETRARO, ITALY, JULY 2003

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Abstract. In the first four lectures we describe a recent proof of the short time existence of curved multidimensional viscous shocks, and the associated justification of the small viscosity limit for piecewise smooth curved inviscid shocks. Our goal has been to provide a detailed, readable, and widely accessible account of the main ideas, while avoiding most of the technical aspects connected with the use of pseudodifferential (or paradifferential) operators. The proof might be described as a combination of ODE/dynamical systems analysis with microlocal analysis, with the main new ideas coming in on the ODE side. In a sense the whole problem can be reduced to the study of certain linear systems of nonautonomous ODEs depending on frequencies as parameters. The frequency-dependent matrices we construct as conjugators or symmetrizers in the process of proving estimates for those ODEs serve as principal symbols of pseudodifferential operators used to prove estimates for the original PDEs.

The linearized problem one has to study in the multiD curved viscous shock problem is one for which there are no available constructive methods (in contrast to the 1D case). In other words we have no idea how to estimate solutions by first constructing them using tools like Fourier-Laplace transforms or Green's functions or even Fourier integral operators and their generalizations. Instead, we rely on energy estimates proved using Kreiss-type symmetrizers. Indeed, our main tool is a symmetrizer for hyperbolic-parabolic boundary problems which generalizes the kind of symmetrizer invented by Kreiss in the early 1970s to deal with hyperbolic boundary problems.

In the final lecture we describe how symmetrizers can be used to study the related (but nonequivalent) problem of long time stability for planar viscous shocks. For zero mass perturbations or nonzero mass perturbations in high space dimensions ($d \geq 5$), one can use symmetrizers just like those used for the first problem (*nondegenerate* symmetrizers). However, in order to get the strongest results by symmetrizer methods (nonzero mass perturbations for dimensions $d \geq 2$) we've had to introduce *degenerate* symmetrizers. In addition, we have to use them in a nonstandard way involving duality and interpolation arguments to get $L^1 - L^p$ estimates instead of L^2 estimates. We'll focus on the use of degenerate symmetrizers in lecture five.

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Date: March 22, 2005.

Thanks to Kevin Zumbrun for a number of suggestions which helped to improve the exposition. Research was supported in part by NSF grant DMS-0070684.

Preface.

These lectures are based on joint work with Olivier Guès, Guy Métivier, and Kevin Zumbrun contained in the papers [GMWZ1, GMWZ2, GMWZ3, GMWZ4] and also on the papers [GW, MZ]. I thank them all for an exciting collaboration that I hope will last well into the future.

There is a large literature dealing with small viscosity limits and long time stability questions for shocks in one space dimension. I'll mention now only some of the papers that deal with questions most closely analogous to the ones studied here. For the small viscosity problem the papers include [GX, MN, R, Y] and for long time stability [Go, KK, L1, L2, LZ, SX, ZH]. In 1D there are also remarkable small viscosity results by quite different methods where the inviscid limits are allowed to be much more general than the piecewise smooth shocks considered here. These results are discussed in the companion lectures by Alberto Bressan.

1. Lecture one: The small viscosity limit: introduction, approximate solution

In this lecture we set up the problem and construct an approximate solution. This construction is one of several places where the inviscid and viscous theories make close contact. In addition, it illuminates part of the role of our main stability hypothesis, and indicates the need to allow variation of the viscous front in the later stability analysis.

We begin with a simple case that still contains most of the main difficulties. Our regularity and hyperbolicity hypotheses can be weakened considerably, and more general viscosities can be treated by these methods (even the degenerate viscosity of compressible Navier-Stokes, we expect; we're checking NS as this is being typed). For those generalizations we refer to the GMWZ papers in the bibliography.

We work in two space dimensions just to make some things easier to write down. The same arguments work in any space dimension.

Consider the $m \times m$ hyperbolic system of conservation laws on $\mathbb{R}^3_{x,t,y}$

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0, \tag{1.1}$$

for which we are given a shock solution $(u^0_\pm, \psi_0)$. This means that u^0_+ (resp. u^0_-) satisfy (1.1) in the classical sense to the right (resp. left) of the shock surface $\mathcal{S}$ defined by $x = \psi_0(t, y)$ and in the distribution sense in a neighborhood of $\mathcal{S}$. The piecewise classical solution is a distribution solution near $\mathcal{S}$ if and only if the Rankine-Hugoniot jump condition holds:

$$\psi^0_t[u^0] + \psi^0_y[g(u^0)] - [f(u^0)] = 0 \text{ on } \mathcal{S}. \tag{1.2}$$

Assumption 1.1 (Hypotheses on the inviscid shock)**.** *(H1) For states u near $u^0_\pm$, the matrix $f'(u)\xi + g'(u)\eta$ has simple real eigenvalues for $(\xi, \eta) \in \mathbb{R}^2 \setminus 0$.*

(H2) The inviscid shock is piecewise smooth (C^∞), exists on the time interval $[0, T_0]$, is constant outside some compact set, and is a Lax shock. This means that the normal matrices

$$A_\nu(u^0_\pm, d\psi_0) \equiv f'(u^0_\pm) - \psi^0_t I - \psi^0_y g'(u^0_\pm) \tag{1.3}$$

are invertible, and that if we let k (resp. l) be the number of positive (resp. negative) eigenvalues of $A_\nu(u^0_+, d\psi^0)$ (resp. $A_\nu(u^0_-, d\psi^0)$), then $k + l = m - 1$.

Consider also a corresponding system of viscous conservation laws on $\mathbb{R}^3_{x,t,y}$

$$\partial_t u + \partial_x f(u) + \partial_y g(u) - \epsilon \triangle u = 0 \tag{1.4}$$

where

$$\triangle = \partial_x^2 + \partial_y^2.$$

A "weak" formulation of the problem we want to consider is: show that the given inviscid shock $(u^0_\pm)$ is the limit in some appropriate sense as $\epsilon \to 0$ of smooth solutions u^ϵ to the parabolic problem (1.4). An appropriate sense of convergence would be, for example, L^2_{loc} near the shock and pointwise away from the shock.

Imagine one had such a family of smooth u^ϵ. In order to have convergence to $u^0_\pm$, there must be a fast transition region located near the inviscid shock $\mathcal{S}$. One might imagine the approximate "center" of that transition region as being defined by a surface $\mathcal{S}^\epsilon$ that approaches $\mathcal{S}$ as $\epsilon \to 0$. If we knew $\mathcal{S}^\epsilon$ we could treat it as an artificial boundary or transmission interface, and proceed to construct u^ϵ by constructing the boundary layers on each side of $\mathcal{S}^\epsilon$ that describe the fast transition.

As a first guess one might take $\mathcal{S}^\epsilon$ to be the known surface $\mathcal{S}$ itself, but that choice turns out to overdetermine the parabolic problem in a sense we'll clarify later. Thus, we are forced to introduce the unknown front $\mathcal{S}^\epsilon$ given by

$$x = \psi^\epsilon(t, y) = \psi^0(t, y) + \epsilon\phi^\epsilon(t, y) \tag{1.5}$$

and to treat ψ^ϵ (or ϕ^ϵ) as an extra unknown (along with u^ϵ). We'll often refer to S^ϵ as *the viscous front*, although as we'll see it is not uniquely determined unless we add an extra condition.

To formulate the transmission problem more clearly, we flatten $\mathcal{S}^\epsilon$ by the change of variables

$$(\tilde{x}, t, y) = (x - \psi^\epsilon(t, y), t, y). \tag{1.6}$$

If we set

$$\tilde{u}^\epsilon(\tilde{x}, t, y) = u^\epsilon(x, t, y) \tag{1.7}$$

and note that $\partial_x u = \partial_{\tilde{x}} \tilde{u}$, $\partial_t u = \partial_t \tilde{u} - \psi_t \partial_{\tilde{x}} \tilde{u}$, $\partial_y u = \partial_y \tilde{u} - \psi_y \partial_{\tilde{x}} \tilde{u}$, we find that the parabolic problem (1.4) in the new variables is (dropping tildes and epsilons)

$$\partial_t u + \partial_x f_\nu(u, d\psi) + \partial_y g(u) - \epsilon \triangle_\psi u = 0 \text{ on } \mathbb{R}^3_{x,t,y}, \tag{1.8}$$

where

$$\begin{aligned} &f_\nu(u, d\psi) = f(u) - \psi_t u - \psi_y g(u) \\ &\triangle_\psi = \partial_x^2 + (\partial_y - \partial_y \psi \partial_x)^2 = ((1 + \psi_y^2)\partial_x^2 - 2\psi_y \partial^2_{xy} + \partial_y^2) - 2\psi_{yy}\partial_x. \end{aligned} \tag{1.9}$$

In the new coordinates the surface $\mathcal{S}^\epsilon$ is $x = 0$. Observe that solving (1.8) is equivalent to solving the transmission problem

$$\begin{aligned} &\partial_t u_\pm + \partial_x f_\nu(u_\pm, d\psi) + \partial_y g(u_\pm) - \epsilon \triangle_\psi u_\pm = 0 \text{ on } \pm x \geq 0, \\ &[u] = 0, [\partial_x u] = 0 \text{ on } x = 0. \end{aligned} \tag{1.10}$$

The transmission problem can easily be reformulated as a standard boundary problem on the half-space $x \geq 0$ by "doubling"; that is, for $x \geq 0$ one can define $\tilde{u}_+(x, t, y) = u_+(x, t, y)$ and $\tilde{u}_-(x, t, y) = u_-(-x, t, y)$. We won't do this yet though, to avoid having to write $\pm$ all the time. In fact, we'll usually write the transmission problem (1.10) without the $\pm$ on u. An important point is that with the transmission formulation we now have tools (like Kreiss-type symmetrizers) from the theory of boundary problems at our disposal to solve the original problem (1.8) on the full space.

Observe that with the extra unknown ψ we should expect the problem (1.10) to be underdetermined and to require an extra boundary condition involving ψ.

1.1. **Approximate solution.**

The first step in solving (1.10) is to construct a high order approximate solution, and the remaining step amounts to proving the stability of that solution. As we explain below uniform stability of the inviscid shock and transversality of the connection play a central part in the construction. The construction also illustrates the importance of allowing variation of the front in the viscous problem.

Since we expect solutions to (1.10) to undergo a fast transition near $x = 0$, it is natural to look for approximate solutions of the form (suppress $\pm$)

$$\begin{aligned} &\tilde{u}^\epsilon(x, t, y) = (\mathcal{U}^0(x, t, y, z) + \epsilon \mathcal{U}^1 + \cdots + \epsilon^M \mathcal{U}^M)|_{z = \frac{x}{\epsilon}} \\ &\tilde{\psi}^\epsilon(t, y) = \psi^0(t, y) + \epsilon \psi^1 + \cdots + \epsilon^M \psi^M, \end{aligned} \tag{1.11}$$

where each profile

$$\mathcal{U}^j(x, t, y, z) = U^j(x, t, y) + V^j(t, y, z) \tag{1.12}$$

is the sum of a slow U^j that describes behavior away from the viscous front and a fast boundary layer profile V^j which decays to 0 (exponentially, it turns out) as $z \to \pm\infty$.

Plug (1.11) into (1.10), collect coefficients of equal powers of ϵ, and separate slow from fast profiles to get

$$\sum_{-1}^{M} \epsilon^j \mathcal{F}^j(x,t,y,z)|_{z=\frac{x}{\epsilon}} + \epsilon^M R^{\epsilon,M}(x,t,y), \tag{1.13}$$

where

$$\mathcal{F}^j(x,t,y,z) = F^j(x,t,y) + G^j(t,y,z) \tag{1.14}$$

and (assuming smooth decaying profiles for the moment),

$$\begin{aligned} &|\partial^\alpha_{t,y}\partial^k_x R^{\epsilon,M}|_{L^\infty} \leq C_{\alpha,k}\epsilon^{-k} \\ &|\partial^\alpha_{t,y}\partial^k_x R^{\epsilon,M}|_{L^2} \leq C_{\alpha,k}\epsilon^{\frac{1}{2}-k}, \end{aligned} \tag{1.15}$$

(caution: each estimate here is two estimates, one in $x \geq 0$ and one in $x \leq 0$). Observe that F^{-1} is automatically zero, and the equations obtained by setting F^0 and G^{-1} equal to zero are

$$\begin{aligned} &(a)\ \partial_t U^0 + \partial_x f_\nu(U^0, d\psi^0) + \partial_y g(U^0) = 0, \\ &(b)\ -(1+(\psi^0_y)^2)\partial_z^2 \mathcal{U}^0 + \partial_z f_\nu(\mathcal{U}^0, d\psi^0) = 0 \end{aligned} \tag{1.16}$$

respectively. Again, each equation here is really two equations; for example, in $\pm z \geq 0$ for (1.16)(b). The coefficient $(1+(\psi^0_y)^2)$ will appear often in what follows; let's call it $B^0(t,y)$.

The equation (1.16)(a) is the inviscid shock problem (1.1) in the new coordinates, so a solution is given by $(U^0_\pm, \psi^0)$, where

$$U^0_\pm(x,t,y) = u^0_\pm(x+\psi^0(t,y),t,y). \tag{1.17}$$

In equation (1.16)(b) $\mathcal{U}^0$ is evaluated at $(0,t,y,z)$ instead of (x,t,y,z). The error of order $O(x)$ introduced by doing this is solved away at the stage of the next fast equation G^0 by writing

$$x = \epsilon\frac{x}{\epsilon} = \epsilon z. \tag{1.18}$$

The boundary conditions in (1.10) yield the boundary profile equations on $x=0$, $z=0$

$$\begin{aligned} &(a)\ U^0_+ + V^0_+ = U^0_- + V^0_- \\ &(b)\ \partial_z V^0_+ = \partial_z V^0_-, \end{aligned} \tag{1.19}$$

or equivalently,

$$[\mathcal{U}^0] = 0,\ [\partial_z \mathcal{U}^0] = 0, \tag{1.20}$$

at the orders ϵ^0, ϵ^{-1} respectively.

Next integrate (1.16)(b) ($\int_{\pm\infty}^z$ in $\pm z \geq 0$) to obtain

$$B^0 \partial_z \mathcal{U}^0_\pm = f_\nu(\mathcal{U}^0_\pm, d\psi^0) - f_\nu(U^0_\pm, d\psi^0), \tag{1.21}$$

where the unknowns are really $V^0_\pm(t,y,z)$, since $U^0_\pm$ are given.

The two boundary conditions in (1.20) clearly overdetermine this first order transmission problem, but note that the Rankine-Hugoniot condition on $(U^0_\pm, d\psi_0)$ is the necessary **compatibility condition**. More precisely, assume that $\mathcal{U}^0$ satisfies (1.21) and $[\mathcal{U}^0]=0$. Then

$$[\partial_z\mathcal{U}^0]=0 \text{ holds } \Leftrightarrow [f_\nu(U^0,d\psi^0)]=0 \text{ on } x=0. \tag{1.22}$$

In view of the transmission conditions (1.20), solving the problem (1.21) for unknowns $V^0_\pm\to 0$ as $z\to\pm\infty$ is equivalent to solving the *connection* problem on $\mathbb{R}_z$ for $W(t,y,z)$

$$\begin{aligned}&(a)B^0\partial_z W=f_\nu(W,d\psi^0)-f_\nu(U^0_-,d\psi^0)\\&(b)W(t,y,z)\to U^0_\pm(0,t,y)=u^0_\pm(\psi^0(t,y),t,y) \text{ as } z\to\pm\infty.\end{aligned}\tag{1.23}$$

From this point of view the Rankine-Hugoniot condition is the statement that $U^0_+(t,y)$ is an equilibrium for the ODE (1.23)(a). We'll refer to the travelling wave equation (1.23)(a) as *the profile equation.* The solution $W(t,y,z)=\mathcal{U}^0(0,t,y,z)$ is variously referred to as a *connection*, a *profile*, and a *viscous shock.* Note that there is a lack of uniqueness due to translation invariance; that is, if $W(t,y,z)$ is a solution, so is $W(t,y,z+a)$ for any $a\in\mathbb{R}$.

Remark 1.1. 1. It is not hard to prove the existence of profiles $W(t,y,z)$ for sufficiently weak Lax shocks (see [MP], for example). For a general discussion of strong shocks we must assume the existence of profiles; however strong shock profiles are known to exist in some specific cases [Gi].

The fact that W decays exponentially to its endstates

$$|W(t,y,z)-U^0_\pm(0,t,y)|=O(e^{-\delta|z|}) \text{ as } z\to\pm\infty \tag{1.24}$$

is a consequence of the invertibility of the normal matrices $A_\nu(U^0_\pm,d\psi^0)$. The latter fact implies $U^0_\pm$ are hyperbolic equilibria for the ODE (1.23).

2. In view of (H2) we see that the range of $W(t,y,z)$ is contained in a compact subset of $\mathbb{R}^m$.

To see how the construction of the higher order profiles works, it will be enough just to consider the case of $(U^1_\pm(x,t,y),d\psi^1(t,y))$ and $V^1_\pm(t,y,z)$. The interior problems satisfied by $V^1_\pm$ are the fast problems at the order ϵ^0. As in the case of $V^0_\pm$, each problem is a second order ODE that can be integrated using the conservative structure to give a first order ODE. The equation for $V^1_\pm$ is a linearization (with respect to both $\mathcal{U}^0_\pm$ and ψ^0) of (1.21) with forcing $Q^0_\pm$ depending on previously determined functions:

$$\begin{aligned}B^0\partial_z V^1_\pm=&A_\nu(\mathcal{U}^0_\pm,d\psi^0)(V^1_\pm+U^1_\pm)-\psi^1_t\mathcal{U}^0_\pm-\psi^1_y g(\mathcal{U}^0_\pm)\\&-\{A_\nu(U^0_\pm,d\psi^0)U^1_\pm-\psi^1_tU^0_\pm-\psi^1_yg(U^0_\pm)\}+Q^0_\pm(t,y,z),\end{aligned}\tag{1.25}$$

where $Q^0_\pm\to 0$ exponentially as $z\to\pm\infty$.

The interior equation for $U^1_\pm$ is a linearization (with respect to $U^0_\pm$) of (1.16)(a):

$$H(U^0_\pm)\partial U^1_\pm:=\partial_tU^1_\pm+A_\nu(U^0_\pm,d\psi^0)\partial_xU^1_\pm+g'(U^0_\pm)\partial_yU^1_\pm=P^0_\pm,\tag{1.26}$$

where again the forcing $P^0_\pm$ depends on previously determined functions.

Again, there are two boundary conditions at $x = 0$, $z = 0$:

$$
(1.27) \qquad \begin{aligned} &(a)\ U^1_+ + V^1_+ = U^1_- + V^1_- \\ &(b)\ \partial_x U^0_+ + \partial_z V^1_+ = \partial_x U^0_- + \partial_z V^1_-, \end{aligned}
$$

so the first order problem for $V^1_\pm$ is overdetermined. Suppose for a moment that (1.25) and (1.27)(a) are satisfied. Then parallel to (1.22) we clearly have the compatibility condition

$$
(1.28) \qquad \begin{aligned} &[\partial_x U^0 + \partial_z V^1] = 0 \Leftrightarrow \text{ on } x = 0, z = 0 \text{ we have} \\ &[A_\nu(U^0, d\psi^0)U^1 - \psi^1_t U^0 - \psi^1_y g(U^0)] = B^0[\partial_x U^0] + [Q^0]. \end{aligned}
$$

Thus, we may arrange the compatibility condition (1.28) by solving the following linearized shock problem for $(U^1_\pm, \psi^1)$:

$$
(1.29) \qquad \begin{aligned} &(a) H(U^0_\pm)\partial U^1_\pm = P^0_\pm(x) \text{ on } \pm x \geq 0 \\ &(b) \psi^1_t[U^0] + \psi^1_y[g(U^0)] - [A_\nu(U^0, d\psi^0)U^1] = -B^0[\partial_x U^0] - [Q^0] \text{ on } x = 0, \end{aligned}
$$

The interior problem (1.29)(a) is the slow problem at the order ϵ^1, and the boundary operator in (b) is a linearization of the Rankine-Hugoniot conditions.

Linearized shock problems like (1.29) were first studied by Majda in [M2] as the first step in his proof of existence of curved multi-D inviscid shocks [M3]. It is a consequence of our main Evans assumption, Assumption (3.1), that the inviscid shock $(U^0_\pm, \psi^0)$ is *uniformly stable* in the sense of Majda [M2]. We'll discuss uniform stability more carefully later, but for now we just state informally that it is essentially equivalent to L^2 well-posedness of problems like (1.29).

So we now have the functions $(U^0_\pm, \psi^0)$, $V^0_\pm$, $(U^1_\pm, \psi^1)$, and the next step is to solve for $V^1_\pm$. We must choose initial data for $V^1_\pm$ at $z = 0$ so that both (1.27)(a) holds and the solution $V^1_\pm$ to (1.25) decays exponentially to 0 as $z \to \pm\infty$. We explain next how a *transversality* condition implied by the same Evans assumption allows us to do this.

Consider again the travelling wave equation (1.23)(a) on $\mathbb{R}_z$, and recall that $U^0_\pm$ are both equilibrium points. Clearly, $W(t, y, z)|_{z=0}$ belongs to both the stable manifold of $U^0_+(0, t, y)$ and the unstable manifold of $U^0_-(0, t, y)$. The Evans assumption implies these manifolds intersect transversally at $W(t, y, 0)$.

For fixed (t, y) let $\mathbb{W}^s(t, y)$ and $\mathbb{W}^u(t, y)$, respectively, be the affine submanifolds of $\mathbb{R}^m$ consisting of initial data at $z = 0$ of solutions to (1.25) that decay as $z \to \pm\infty$. These submanifolds are translates of the tangent spaces (at $W(t, y, 0)$) to the above stable and unstable manifolds, so they too intersect transversally. Equivalently, the intersection of the affine submanifolds

$$
(1.30) \quad (\mathbb{W}^s(t, y) \times \mathbb{W}^u(t, y)) \cap \{(v_1, v_2) \in \mathbb{R}^{2m} : v_1 - v_2 = U^1_-(0, t, y) - U^1_+(0, t, y)\}
$$

is transversal, hence nonempty. In fact, since by assumption (H2) the dimension of $\mathbb{W}^s(t, y) \times \mathbb{W}^u(t, y)$ is $m + 1$, the intersection (1.30) is a line. Thus, we obtain a one-parameter family of choices of initial data for decaying solutions of (1.25) satisfying (1.27)(a).

We continue according to this pattern to solve for $(U^2_\pm,\psi^2)$, then $V^2_\pm$, then $(U^3_\pm,\psi^3)$, etc., always obtaining linearized Majda well-posed shock problems for $(U^j_\pm,\psi^j)$ whose boundary conditions are chosen as the compatibility conditions for the overdetermined problems satisfied by $V^j_\pm$.

Remark 1.2. 1. Later we'll add an extra boundary condition (2.10), and one effect of this will be to remove the nonuniqueness in the higher profiles.

2. The boundary condition (1.27)(b) shows that in general $[\partial_z V^1]\neq 0$, so one can't solve for $V^1_\pm$ by solving a single ODE on $\mathbb{R}_z$ as we did for $V^0_\pm$.

3. The above construction doesn't work if one simply fixes $\tilde\psi^\epsilon=\psi^0$. If one does not allow the variation in the front given by ψ^1, for example, the problem (1.29) is overdetermined and generally unsolvable. A similar statement applies to ψ^j for $j>1$.

1.2. **Summary.** Let's write the transmission problem (1.10) as

$$\begin{aligned}&\mathcal{E}(u,\psi)=0\\&[u]=0,[\partial_x u]=0.\end{aligned}\tag{1.31}$$

Recalling (1.13) we now have an approximate solution $(\tilde u,\tilde\psi)$ defined on a fixed time interval independent of ϵ (determined by the time of existence of the given inviscid shock) such that

$$\begin{aligned}&\mathcal{E}(\tilde u,\tilde\psi)=\epsilon^M R^{\epsilon,M}\\&[\tilde u]=0,[\partial_x \tilde u]=0.\end{aligned}\tag{1.32}$$

We proceed to look for an exact solution to (1.31) of the form

$$u=\tilde u+v,\psi=\tilde\psi+\phi.\tag{1.33}$$

The main difficulty is to obtain good L^2 estimates for the linearization of (1.31) about $(\tilde u,\tilde\psi)$. Once these are in hand it is fairly routine to obtain higher derivative estimates by differentiating the equation, and to then solve the error equation for (v,ϕ) by Picard iteration (i.e., contraction).

Here is the main result:

Theorem 1.1. *Under assumptions* (1.1) *and* (3.1) *there exists an* $\epsilon_0>0$ *such that for* $0<\epsilon\leq\epsilon_0$ *the parabolic transmission problem* (1.10) *has an exact solution on* $[0,T_0]\times\mathbb{R}^2_{x,y}$ *of the form*

$$u^\epsilon=\tilde u+v,\ \psi^\epsilon=\tilde\psi+\phi,\tag{1.34}$$

where $(\tilde u,\tilde\psi)$ *is an approximate solution satisfying* (1.32). *For arbitrary positive integers* K *and* L*, provided* $M=M(K,L)$ *in* (1.32) *is taken large enough, we have the estimates*

$$\begin{aligned}&|\partial^\alpha(v,\epsilon\partial_x v)|_{L^2(x,t,y)}+|\partial^\alpha(v,\epsilon\partial_x v)|_{L^\infty(x,t,y)}\leq\epsilon^L\quad(\partial=\partial_{t,y})\\&|\partial^\alpha\phi|_{L^2(t,y)}+|\partial^\alpha\phi|_{L^\infty(t,y)}\leq\epsilon^L\end{aligned}\tag{1.35}$$

for $|\alpha|\leq K$.

Remark 1.3. The theorem asserts the stability of the boundary layer given by the approximate solution, and allows us to read off a precise sense in which the solution u^ϵ of (1.4) converges to the inviscid shock u^0. For example, we have convergence in L^2_{loc} near the shock, and in C^0_{loc} away from the shock.

2. Lecture two: Full linearization, reduction to ODEs, conjugation to a limiting problem

2.1. **Full versus partial linearization.** To find the error problem satisfied by (v, ϕ) we first rewrite (1.31)

$$\mathcal{E}(\tilde{u} + v, \tilde{\psi} + \phi) = \mathcal{E}(\tilde{u}, \tilde{\psi}) + \mathcal{E}'_u(\tilde{u}, \tilde{\psi})v + \mathcal{E}'_\psi(\tilde{u}, \tilde{\psi})\phi + Q(v, \phi) = 0, \tag{2.1}$$

where $\mathcal{E}'_u$ and $\mathcal{E}'_\psi$ are the linearizations of $\mathcal{E}$ with respect to u and ψ respectively, and Q is a sum of terms at least quadratic in $\partial^\alpha(v, \phi)$, $|\alpha| \leq 2$.

Thus, we must solve the transmission problem

$$\begin{aligned} &\mathcal{E}'_u(\tilde{u}, \tilde{\psi})v + \mathcal{E}'_\psi(\tilde{u}, \tilde{\psi})\phi = -\epsilon^M R^{\epsilon,M} - Q(v, \phi) \\ &[v] = [\partial_x v] = 0. \end{aligned} \tag{2.2}$$

The explicit formulas for the linearizations are

$$\begin{aligned} &\mathcal{E}'_u(\tilde{u}, \tilde{\psi})v = \partial_t v + \partial_x(A_\nu(\tilde{u}, \tilde{\psi})v) + \partial_y(g'(\tilde{u})v) - \epsilon\triangle_{\tilde{\psi}} v \\ &\mathcal{E}'_\psi(\tilde{u}, \tilde{\psi})\phi = -\phi_t\partial_x\tilde{u} - \phi_y\left(g'(\tilde{u})\partial_x\tilde{u} - 2\epsilon(\partial_y - \tilde{\psi}_y\partial_x)\partial_x\tilde{u}\right) + \epsilon\phi_{yy}\partial_x\tilde{u}. \end{aligned} \tag{2.3}$$

One should expect there to be some simple relationship between the two operators in (2.3). Much of what follows (both in this and later sections) hinges on observing that

$$\mathcal{E}'_\psi(\tilde{u}, \tilde{\psi})\phi = -\mathcal{E}'_u(\tilde{u}, \tilde{\psi})(\phi\partial_x\tilde{u}) + \phi\partial_x(\mathcal{E}(\tilde{u}, \tilde{\psi})). \tag{2.4}$$

This can be proved by a direct verification; later we'll see that it becomes rather obvious after a few reductions.

This implies that the left side of (2.2) is the same as

$$\mathcal{E}'_u(\tilde{u}, \tilde{\psi})(v - \phi\partial_x\tilde{u}) + \phi\partial_x(\mathcal{E}(\tilde{u}, \tilde{\psi})),$$

so we reduce to solving

$$\mathcal{E}'_u(\tilde{u}, \tilde{\psi})(v - \phi\partial_x\tilde{u}) = -\epsilon^M R^{\epsilon,M} - Q(v, \phi) - \phi\partial_x(\epsilon^M R^{\epsilon,M}). \tag{2.5}$$

This suggests the strategy of reducing the study of the fully linearized operator given by the left side of (2.2) to that of the partially linearized operator $\mathcal{E}'_u$ by introducing the "good unknown"

$$v^\# = v - \phi\partial_x\tilde{u}. \tag{2.6}$$

Indeed this strategy turns out to work well in what we'll soon define as the medium and high frequency regimes, where $\mathcal{E}'_u$ is nonsingular. In the low frequency regime, we'll see that $\mathcal{E}'_u$ is singular, but that the singularity can be removed by the introduction of an extra boundary condition and a more subtle choice of "good unknown".

It is really just in the low frequency estimates that we make essential use of a modified unknown depending on the front. In the medium and high frequency regimes the unknown $v^\#$ serves to remove complications due to the variable ϕ which is somewhat artificial in those regimes.

Remark 2.1. The need to consider the full linearization is not obvious at this point. We have a high order approximate solution $(\tilde{u}, \tilde{\psi})$, so why not just fix $\tilde{\psi}$ once and for all, and solve (1.31) by solving

$$\begin{aligned} &\mathcal{E}(\tilde{u}+v, \tilde{\psi}) = 0 \\ &[v] = 0, [\partial_x v] = 0 \end{aligned} \tag{2.7}$$

for v. Indeed, this was the strategy pursued in [GMWZ2]. What happens is that the singularity in $\mathcal{E}'_u$ in the low frequency regime leads to a slightly degenerate linearized L^2 estimate of the form

$$\sqrt{\epsilon}|v|_{L^2} \leq |F|_{L^2}, \tag{2.8}$$

provided we restrict how much the curved inviscid shock $\mathcal{S}$ can deviate from flatness. The estimate (2.8) somewhat surprisingly turns out to be good enough for Picard iteration, and this yields a solution of (1.31).

But, of course, it is highly desirable to remove the restriction on how much $\mathcal{S}$ can curve, and we'll be able to do that by adding an extra boundary condition and working with the full linearization. That way we'll get an estimate without the $\sqrt{\epsilon}$ on the left as in (2.8).

2.2. **The extra boundary condition.** Even though an extra boundary condition is not strictly needed except in the low frequency regime, where the unknown $v^\#$ in (2.6) is not used, we can use (2.5) to motivate our choice of boundary condition.

If we work with the full linearization we'll need estimates on ϕ in all frequency regimes, so the idea is to choose a boundary condition that will yield estimates for ϕ once we have an estimate on the trace of $v^\#$. The function ϕ is scalar so we choose a vector $l(t, y)$ and consider on $x = 0$

$$l \cdot v^\# = l \cdot v - \phi(l \cdot \partial_x \tilde{u}). \tag{2.9}$$

The leading part of $\partial_x \tilde{u}$ is $\frac{1}{\epsilon}\partial_z W$, so if we choose $l(t, y)$ so that $l(t, y)\cdot\partial_z W(t, y, 0) = -1$ and demand, for example, that

$$l \cdot v = \partial_t \phi - \epsilon \partial_y^2 \phi \text{ on } x = 0, \tag{2.10}$$

the leading part of the right side of (2.9) is

$$\partial_t \phi - \epsilon \partial_y^2 \phi + \frac{\phi}{\epsilon}.$$

Set $\tilde{\phi} = \frac{\phi}{\epsilon}$ and rewrite (2.9) to obtain

$$l \cdot v^\# = (\epsilon \partial_t - \epsilon^2 \partial_y^2 + 1)\tilde{\phi}, \tag{2.11}$$

which leads to an estimate for $\tilde{\phi}$ if one has an estimate for the trace of $v^\#$. Other similar choices are of course possible for the differential operator in (2.10), but we'll settle on (2.10) as the extra boundary condition. This condition also turns out to work well in the low frequency regime. So we must solve the nonlinear transmission problem

$$
\begin{aligned}
&\mathcal{E}'_u(\tilde{u},\tilde{\psi})v + \mathcal{E}'_\psi(\tilde{u},\tilde{\psi})\phi = -\epsilon^M R^{\epsilon,M} - Q(v,\phi) \\
&[v] = [\partial_x v] = 0, \partial_t\phi - \epsilon\partial_y^2\phi - l\cdot v = 0 \text{ on } x = 0.
\end{aligned} \tag{2.12}
$$

Remark 2.2. One can also impose the extra boundary condition in (1.31), before even constructing the approximate solution or defining (v,ϕ). Here a good choice is

$$
\partial_t\psi - \epsilon\partial_y^2\psi - l\cdot u = \partial_t\psi^0 - \epsilon\partial_y^2\psi^0 - l\cdot W(t,y,0). \tag{2.13}
$$

It's easy to check that imposing this extra condition in the construction of the approximate solution $(\tilde{u},\tilde{\psi})$ removes the translational nonuniqueness that we observed for every fast profile $V^j_\pm(t,y,z)$. They are now pinned down. If one takes that uniquely determined solution $(\tilde{u},\tilde{\psi})$ and defines (v,ϕ) as we did above, the extra condition on (v,ϕ) is exactly that in (2.12).

2.3. **Corner compatible initial data and reduction to a forward problem.** To determine a unique solution of (2.12) we must impose some initial conditions

$$
v = \epsilon^{M_0} v_0, \phi = \epsilon^{M_0}\phi_0 \text{ at } t = 0. \tag{2.14}
$$

We are free to choose these in many different ways, but our later work will be much easier if we choose the initial data to satisfy *corner compatibility* conditions to high order at the corner $x = 0, t = 0$ (recall, we can also formulate (2.12) as a doubled boundary problem, and then we really do have a corner). We show how to do that in [GW] or [GMWZ2], but those rather technical details seem like good ones to omit here. The key point is that such data will allow us to obtain solutions to the transmission problem that are piecewise smooth to high order in $\pm x \geq 0$ (recall the interior forcing term $R^{\epsilon,M}$ is piecewise smooth). The M_0 in (2.14) is about $M - k_0$, where k_0 is the order of compatibility. It's worth noting that the high order approximate solution plays an important role here. Arranging compatibility uses x derivatives, and each x derivative introduces a power of $\frac{1}{\epsilon}$. This leads to the reduction in M.

This choice of initial data allows us to reduce to a forward transmission problem, one where all data is zero in $t < 0$, and the boundary conditions are homogeneous. By a standard maneuver from linear PDE, we first transfer initial data to forcing at the price of introducing nonzero boundary forcing. But the above compatibility conditions are designed precisely so that the new boundary forcing vanishes to high order at $t = 0$. Next a similar maneuver transfers the nonzero boundary forcing to interior forcing, leaving us with the forward error problem (relabelling unknowns (v,ϕ) again) in $\mathbb{R}^3_{x,t,y}$

$$
\begin{aligned}
&\mathcal{E}'_u(\tilde{u},\tilde{\psi})v + \mathcal{E}'_\psi(\tilde{u},\tilde{\psi})\phi = -\epsilon^{M'} F - Q(v,\phi) \\
&[v] = [\partial_x v] = 0, \partial_t\phi - \epsilon\partial_y^2\phi - l\cdot v = 0 \text{ on } x = 0 \\
&v = 0, \phi = 0 \text{ in } t < 0,
\end{aligned} \tag{2.15}
$$

where $F = 0$ in $t < 0$ and piecewise smooth to high order in $\pm x \geq 0$. Here $M' < M$, but M' is still large provided M was large compared to k_0. Q is not the same as before, but the earlier properties attributed to Q continue to hold.

Remark 2.3. We're free to introduce a cutoff in time on the right in (2.15), so we can choose extensions of $\tilde{u}$ and $\tilde{\psi}$ to all time so that the coefficients in (2.15) are defined for all time.

2.4. **Principal parts, exponential weights.** The main task now is to prove a nondegenerate L^2 estimate for the fully linearized forward transmission problem on $\mathbb{R}^3_{x,t,y}$

$$\begin{aligned} &\mathcal{E}'_u(\tilde{u},\tilde{\psi})v + \mathcal{E}'_\psi(\tilde{u},\tilde{\psi})\phi = f \\ &[v] = 0, [\partial_x v] = 0, \partial_t\phi - \epsilon\partial_y^2\phi - l\cdot v = 0 \text{ on } x = 0 \\ &v = 0, \phi = 0, f = 0 \text{ in } t < 0. \end{aligned} \tag{2.16}$$

Recall that the leading parts of $\tilde{u}$ and $\tilde{\psi}$ are $W(t,y,z)|_{z=\frac{x}{\epsilon}}$ and $\psi^0(t,y)$ respectively. If we replace $(\tilde{u},\tilde{\psi})$ by $(W(t,y,\frac{x}{\epsilon}),\psi^0(t,y))$ in (2.16), we introduce several error terms in the estimate, including some of size

$$O(|v|_{L^2}) + O(|\epsilon\partial_x v|_{L^2}) + O(|d\phi|_{L^2}) + O(\epsilon|d^2\phi|_{L^2}). \tag{2.17}$$

Errors like (2.17) turn out to be absorbable by the left side of the estimate we obtain.

Remark 2.4. The difference $U^0(x,t,y) - U^0(0,t,y)$ leads to another sort of error when $\tilde{u}$ is replaced by W that can't simply be absorbed as above. When $|x|$ is small though, the coefficients are perturbed only slightly by this difference, so the symmetrizer construction presented below for the slightly simplified case where this perturbation is ignored works in the same way for the case where the perturbation is included.

Thus, the proof of the L^2 estimate for the original linearized PDE needs to be split in two parts; one where the solution is supported near the boundary, and another for solutions supported away from the boundary (errors introduced by cutoffs used to localize the estimate are again absorbable). In the second case there are no boundary conditions, no glancing modes, and no singular terms in the linearized operator, and the estimates can be proved by a much simpler argument (for details we refer to Propositions 5.6 and 5.7 of [MZ]). We focus on the symmetrizer construction needed for the estimates near the boundary in these notes.

There are other terms like $\partial_y(g'(W))v$ in (2.16) of size (2.17). Throwing away all such terms in (2.3) and replacing $(\tilde{u},\tilde{\psi})$ by (W,ψ^0) we obtain the *principal parts*

$$\begin{aligned} &\mathcal{L}_u(t,y,\frac{x}{\epsilon},\partial_{x,t,y})v = \\ &\partial_t v + A_\nu(W,d\psi^0)\partial_x v + g'(W)\partial_y v + \frac{(d_u A_\nu\cdot W_z)v}{\epsilon} - \epsilon\left(B^0\partial_x^2 + \partial_y^2 - 2\psi_y^0\partial_{xy}^2\right)v, \\ &\mathcal{L}_\psi(t,y,\frac{x}{\epsilon},\partial_{t,y})\phi = -\frac{1}{\epsilon}\left(W_z\phi_t + g'(W)W_z\phi_y + 2\psi_y^0 W_{zz}\phi_y - \epsilon W_z\phi_{yy}\right). \end{aligned} \tag{2.18}$$

At this point we introduce exponential weights in time $e^{-\gamma t}$, $\gamma \geq 1$ and define

$$v_\gamma = e^{-\gamma t}v, \phi_\gamma = e^{-\gamma t}\phi, f_\gamma = e^{-\gamma t}f. \tag{2.19}$$

If (v, ϕ) satisfies

$$\mathcal{L}_u v + \mathcal{L}_\psi \phi = f \tag{2.20}$$

with the above boundary conditions, then (v_γ, ϕ_γ) satisfies the same problem with ∂_t replaced by $\partial_t + \gamma$

$$\begin{aligned} &\mathcal{L}_u(t, y, \frac{x}{\epsilon}, \partial_t + \gamma, \partial_{x,y})v_\gamma + \mathcal{L}_\psi(t, y, \frac{x}{\epsilon}, \partial_t + \gamma, \partial_y)\phi_\gamma = f_\gamma \\ &[v_\gamma] = 0, [\partial_x v_\gamma] = 0, (\partial_t + \gamma)\phi_\gamma - \epsilon\partial_y^2\phi_\gamma - l \cdot v_\gamma = 0 \text{ on } x = 0 \\ &v_\gamma = 0, \phi_\gamma = 0, f_\gamma = 0 \text{ in } t < 0. \end{aligned} \tag{2.21}$$

Henceforth, we'll drop the subscripts γ, but the exponential weights are always there. Also, we'll often neglect to mention that all data is zero in the past.

Remark 2.5. Why introduce exponential weights? One reason is that γ will be an important parameter in the later stability analysis, but that is hardly apparent now. Another reason is that we'll sometimes be able to absorb errors by taking γ large (not apparent now). Another is that later we'll freeze (t, y) in the coefficients, and it's common to take a Fourier-Laplace transform in time, thereby introducing an exponential weight, in the stability analysis of constant coefficient problems (not convincing yet, perhaps). Finally, a reason that seems clear even at this point is that it is technically much more convenient to estimate solutions to (2.21) on $\mathbb{R}^3_{x,t,y}$ instead of on a bounded time domain (e.g., we'll use pseudodifferential operators acting in (t, y)), and one can't expect solutions of (2.20) to have bounded L^2 norms for all time, but one can expect that to be true for (2.21) if γ is large enough. The reader is invited to check this last point by looking, for example, at explicit solutions of the scalar convection equation

$$u_t + au_x + bu_y = f. \tag{2.22}$$

2.5. **Some difficulties.** We have to find estimates for (2.21) that are uniform in ϵ as $\epsilon \to 0$, but a quick glance at $\mathcal{L}_u$ and $\mathcal{L}_\psi$ does not encourage optimism on that point. On the one hand the coefficients contain a mixture of powers of ϵ, including ϵ^{-1}. In addition, the crucial normal matrix A_ν, although nonsingular at the endstates $U^0_\pm(0, t, y)$, is singular at some intermediate value of $z = \frac{x}{\epsilon}$ since one of its eigenvalues changes sign along the profile (the pth eigenvalue if the inviscid shock is a p shock). Indeed, the latter problem is one of the main difficulties in the entire analysis. Another difficulty we should expect from experience with hyperbolic boundary problems is that, since we are working in multiD we'll have to contend with *glancing points*. We'll define these later but for now we just point out that they correspond to characteristics that are tangent to the free boundary given by the curved viscous front $\mathcal{S}^\epsilon$. Except for special cases it is not known how to construct explicit solutions (using Green's functions, parametrices, Fourier integral operators, ...) for the linearized problem near glancing points. In the case of planar fronts the Fourier-Laplace transform can be used. In the case of first-order tangency the Fourier-Airy integral operators of Melrose and Taylor are available. For higher order tangency there are no constructive methods as far as we know. Fortunately, we will be able to construct Kreiss-type symmetrizers to deal with glancing points of any order. Indeed, one of the main points of our work is that, even in these singular hyperbolic-parabolic problems, one can avoid explicit constructions by using symmetrizers much as Kreiss did in the early 1970s for hyperbolic boundary problems.

2.6. **Semiclassical form.** The equations become more balanced in ϵ if we simply multiply through by epsilon and write them in semiclassical form.

Let (τ,η) be dual variables to (t,y) (in the Fourier transform sense), and set $\zeta = (\tau,\gamma,\eta)$ and $\tilde{\zeta} = \epsilon\zeta$. Define semiclassical symbols

$$(2.23)\quad \begin{aligned} &(a) \qquad L_u(t,y,z,\tilde{\zeta},\partial_z) = \\ &\big((i\tilde{\tau}+\tilde{\gamma}+\tilde{\eta}^2)I + g'(W)i\tilde{\eta} + d_uA_\nu\cdot W_z\big) + \big(A_\nu(W,d\psi^0)+2\psi^0_y i\tilde{\eta}\big)\,\partial_z - B^0\partial_z^2 \\ &(b) \qquad L_\psi(t,y,z,\tilde{\zeta}) = -\big(W_z(i\tilde{\tau}+\tilde{\gamma}+\tilde{\eta}^2) + g'(W)W_z i\tilde{\eta} + 2\psi^0_y W_{zz} i\tilde{\eta}\big) \\ &(c) \qquad p(\tilde{\zeta}) = (i\tilde{\tau}+\tilde{\gamma}+\tilde{\eta}^2). \end{aligned}$$

Then if we set $\tilde{f} = \epsilon f$, $\tilde{\phi} = \frac{\phi}{\epsilon}$, $(D_t,D_y) = \frac{1}{i}(\partial_t,\partial_y)$, and change variables $z = \frac{x}{\epsilon}$, we can rewrite (2.21) in semiclassical form (dropping subscripts γ)

$$(2.24)\quad \begin{aligned} &L_u(t,y,z,\epsilon D_t,\epsilon\gamma,\epsilon D_y,\partial_z)v + L_\psi(t,y,z,\epsilon D_t,\epsilon\gamma,\epsilon D_y)\tilde{\phi} = \tilde{f} \\ &[v]=0,\ [\partial_z v]=0,\ p(\epsilon D_t,\epsilon\gamma,\epsilon D_y)\tilde{\phi} - l\cdot v = 0 \text{ on } z=0. \end{aligned}$$

Here we've used $\partial_z = \epsilon\partial_x$.

2.7. **Frozen coefficients; ODEs depending on frequencies as parameters.** Set $q=(t,y)$ and parallel to (2.24), consider on $\mathbb{R}_z$ the system of transmission ODEs depending on parameters $(q,\tilde{\zeta})$:

$$(2.25)\quad \begin{aligned} &L_u(q,z,\tilde{\zeta},\partial_z)v + L_\psi(q,z,\tilde{\zeta})\phi = f \\ &[v]=0,\ [\partial_z v]=0,\ p(\tilde{\zeta})\phi - l\cdot v = 0 \text{ on } z=0. \end{aligned}$$

Here $v=v(z)$, $f=f(z)$, and ϕ is a scalar unknown.

Observe that if we freeze the variables $(t,y)=q$ in (2.24) and then take the Fourier transform in (t,y), we arrive at a problem just like (2.25) with

$$(2.26)\quad v(z) = \widehat{v}(\tau,\eta,z),\quad \phi = \widehat{\tilde{\phi}}(\tau,\eta),\quad f(z) = \widehat{\tilde{f}}(\tau,\eta,z).$$

Here the hat denotes Fourier transform and the absence of tildes on (τ,η) is correct in (2.26). Thus, there are two paths, equivalent of course, to the system of ODEs (2.25).

Consider the problem of proving estimates, uniform in the parameters $(q,\tilde{\zeta})$, for the system of transmission ODEs (2.25). Here we refer to estimates of

$$(2.27)\quad |v|_2 := |v|_{L^2(z)},\ |\phi| := |\phi|_{\mathbb{C}},$$

weighted by appropriate functions depending on the frequency, in terms of $|f|_2$. An immediate consequence of Plancherel's theorem and (2.26) is that such estimates imply corresponding estimates for the semiclassical system of PDEs (2.24) with coefficients frozen at $(t,y)=q$, and hence (after unravelling the changes of variables) estimates on the $|v|_{L^2(x,t,y)}$ and $|\partial_{t,y}\phi, \gamma\phi|_{L^2(t,y)}$ norms of solutions to the *frozen* version of (2.21).

Now it turns out, rather remarkably, that the same constructions needed (by us) to prove uniform estimates for the system of ODEs (2.25) are also the main steps in the proof of estimates even for the *variable coefficient* problem (2.21). To really see how this can be, one needs to use pseudodifferential (or paradifferential) operators and Garding inequalities, and we'll say a bit about those later. The point is that constructions for the ODEs (constructions of objects like conjugators, symmetrizers, ...) are exactly the same as the constructions of the *principal symbols* of the pseudodifferential operators that we use to prove estimates for the variable coefficient system of PDEs (2.21). So it's not too much of a stretch to say that if one takes pseudodifferential calculus as a given, the problem of proving estimates uniform with respect to frequency for the ODEs (2.25) is essentially equivalent to the problem of proving estimates uniform in epsilon for (2.21). We stress this because much of our effort from now on will be devoted to understanding the ODEs (2.25).

Remark 2.6. The relationship we observed earlier between the linearizations $\mathcal{E}'_u$ and $\mathcal{E}'_\psi$ (2.4) is more obvious now at the ODE level. We have

$$L_u(q, z, \tilde{\zeta}, \partial_z)W_z = -L_\psi(q, z, \tilde{\zeta}) \tag{2.28}$$

This is clear by inspection of (2.23) or can be deduced by looking at the leading $O(\frac{1}{\epsilon^2})$ term of (2.4). Observe that the leading part of $\partial_x(\mathcal{E}(\tilde{u}, \tilde{\psi}))$ in (2.4) is zero, since W satisfies the profile equation.

2.8. **Three frequency regimes.** Consider the frequency $\tilde{\zeta} = \epsilon(\tau, \gamma, \eta)$ that appears in (2.25). Estimates of solutions to (2.25) depend critically on the size of $|\tilde{\zeta}|$. The small, medium, and large frequency regimes (SF,MF,HF) are respectively

$$|\tilde{\zeta}| \leq \delta, \; \delta \leq |\tilde{\zeta}| \leq R, \; |\tilde{\zeta}| \geq R \tag{2.29}$$

for small enough δ and large enough R to be determined.

The compact set of nonzero medium frequencies, where $L_u(q, z, \tilde{\zeta}, \partial_z)$ is nonsingular, is the easiest to handle. SF, where L_u is singular and hyperbolic and parabolic effects mix in a subtle way, is the hardest by far. In HF parabolic effects dominate and the hyperbolic part behaves like a perturbation, but care is needed because frequencies occur with mixed homogeneities and vary in an unbounded set.

Caution: When we speak of small, medium, or large frequencies, we are referring to the size of $\tilde{\zeta} = \epsilon\zeta$, not $|\zeta|$. For example, $|\zeta|$ can be extremely large in the small frequency regime, provided ϵ is small enough. *In our analysis of* (2.25) *we're going to drop the tilde on* ζ *from now on*, but it's important to remember the tilde is there before translating back to results for the PDE (2.21).

2.9. **First-order system.** To prepare the way for conjugation and the construction of symmetrizers let's rewrite (2.25) as a $2m \times 2m$ first order system for the unknown

(U, ϕ). Setting $U = (v, \partial_z v)$, we have

$$\begin{aligned} &\partial_z U - G(q,z,\zeta)U = F + \begin{pmatrix} 0 \\ -(B^0(q))^{-1} L_\psi(q,z,\zeta)\phi) \end{pmatrix} \\ &[U] = 0,\ p(\zeta)\phi - l(q)\cdot v = 0 \text{ on } z = 0 \end{aligned} \tag{2.30}$$

where

$$\begin{aligned} F &= \begin{pmatrix} 0 \\ -(B^0(q))^{-1} f \end{pmatrix} \text{ and} \\ G(q,z,\zeta) &= \begin{pmatrix} 0 & I \\ \mathcal{M} & \mathcal{A} \end{pmatrix}, \end{aligned} \tag{2.31}$$

with

$$\begin{aligned} &\mathcal{M}(q,z,\zeta) = (B^0(q))^{-1}\left((i\tau + \gamma + \eta^2)I + g'(W)i\eta + d_u A_\nu \cdot W_z\right), \\ &\mathcal{A} = (B^0(q))^{-1}\left(A_\nu(W, d\psi^0) + 2\psi^0_y i\eta\right). \end{aligned} \tag{2.32}$$

For now we'll ignore the L_ψ term in (2.30) and study the problem

$$\partial_z U - G(q,z,\zeta)U = F,\ [U] = 0. \tag{2.33}$$

Since $W \to U^0_\pm(q)$ as $z \to \pm\infty$ we also have the limiting systems on $\pm z \geq 0$

$$\partial_z V - G_{\pm\infty}(q,\zeta)V = \tilde{F}, \tag{2.34}$$

where

$$G_{\pm\infty}(q,\zeta) = \begin{pmatrix} 0 & I \\ \mathcal{M}_{\pm\infty} & \mathcal{A}_{\pm\infty} \end{pmatrix} \tag{2.35}$$

with

$$\begin{aligned} &\mathcal{M}_{\pm\infty}(q,\zeta) = (B^0(q))^{-1}\left((i\tau + \gamma + \eta^2)I + g'(U^0_\pm(q))i\eta\right) \\ &\mathcal{A}_{\pm\infty} = (B^0(q))^{-1}\left(A_\nu(U^0_\pm(q), d\psi^0) + 2\psi^0_y i\eta\right). \end{aligned} \tag{2.36}$$

Here we've written $U^0_\pm(q)$ for $U^0_\pm(0,q)$.

2.10. **Conjugation.** It would be a great simplification to reduce the study of the variable coefficient problem (2.33) to that of the constant coefficient problem (2.34) (with an appropriately altered boundary condition), and we proceed to do that now in SF and MF but not HF. Since A_ν is nonsingular at the endstates $U^0_\pm(q)$ but not along $W(q,z)$, we'll then be in a much better position to construct symmetrizers and prove estimates.

There are classical results in the theory of ODEs (see, e.g. [Co]) that establish a correspondence between solutions to variable coefficient ODEs and solutions to corresponding limiting constant coefficient ODEs, when those limits exist and satisfy certain hypotheses. For such a correspondence to be useful here, we need it to be somehow uniform in the parameters (q, ζ). However, even if we restrict ζ to a compact set, the classical results fail to apply near $\zeta = 0$ because of the degeneracy

$$G_{\pm\infty}(q,0) = \begin{pmatrix} 0 & I \\ 0 & \mathcal{A}_{\pm\infty}(q,0) \end{pmatrix}. \tag{2.37}$$

Nevertheless, using the Gap Lemma of [GZ, KS] we can, locally near any basepoint $(\underline{q}, \underline{\zeta})$, reduce the study of (2.33) to (2.34) by constructing matrices $Z_\pm(q, z, \zeta)$ as in the following Lemma.

Lemma 2.1. *There exist matrices $Z_\pm(q, z, \zeta)$ in $\pm z \geq 0$ depending smoothly on (q, z, ζ) such that*

$$\begin{aligned}&(a)\partial_z Z = GZ - ZG_{\pm\infty} \text{ on } \pm z \geq 0\\ &(b)|Z(q, z, \zeta) - I| = O(e^{-\theta|z|}) \text{ for some } \theta > 0\\ &(c)Z^{-1}(q, z, \zeta) \text{ is uniformly bounded for } (q, \zeta) \text{ near } (\underline{q}, \underline{\zeta}),\ \pm z \geq 0.\end{aligned} \tag{2.38}$$

Suppose for a moment that we have such $Z_\pm$. Set $U = ZV$. Then expanding out $\partial_z(ZV)$ shows that U satisfies (2.33) if and only if V satisfies

$$\begin{aligned}&\partial_z V - G_{\pm\infty} V = Z^{-1} F \text{ on } \pm z \geq 0,\\ &[ZV] = 0 \text{ on } z = 0.\end{aligned} \tag{2.39}$$

The acceptable price is a more complicated boundary condition. When we translate back to the PDE problem, the new boundary condition is pseudodifferential. The properties of Z in (2.38) show that L^2 estimates for (2.39) can be immediately transported to L^2 estimates for (2.33).

To construct Z_+ say, note that the matrix ODE (2.38)(a) can be written

$$\partial_z Z = \mathcal{L} Z + (\Delta G) Z \tag{2.40}$$

where $\mathcal{L}(q, \zeta)$ is the constant coefficient operator given by the commutator $[G_{+\infty}, \cdot]$ and ΔG is left multiplication by $G - G_{+\infty} = O(e^{-\delta z})$ (δ as in Remark 1.1).

The ODE (2.40) also has a limiting problem $\partial_z Y = \mathcal{L} Y$. The identity matrix I is an eigenvector of $\mathcal{L}$ associated to the eigenvalue 0, and hence $Y(z) = I$ solves the limiting problem. Suppose that the eigenvalues of $\mathcal{L}(q, \zeta)$, which are differences of eigenvalues of $G_{+\infty}(q, \zeta)$, avoid a line $\Re\mu = -\kappa$ for some $0 < \kappa < \delta$. This will always be true for (q, ζ) close enough to a fixed basepoint $(\underline{q}, \underline{\zeta})$.

A solution of (2.40) close to the solution I of $\partial_z Y = \mathcal{L} Y$ could then be found by solving the equation on $z \geq 0$

$$Z_+(q, z, \zeta) = I + \int_0^z e^{(z-s)\mathcal{L}} \pi_-(\Delta G)(s) Z_+(s) ds - \int_z^{+\infty} e^{(z-s)\mathcal{L}} \pi_+(\Delta G)(s) Z_+(s) ds, \tag{2.41}$$

where $\pi_\pm(q, \zeta)$ are the spectral projectors on the generalized eigenspaces of $\mathcal{L}$ corresponding to eigenvalues with $\Re\mu > -\kappa$, $\Re\mu < -\kappa$ respectively. Note that the range of π_+ includes part of the *negative* eigenspace of $\mathcal{L}$, so we might expect the second integral in (2.41) to blow up since $z - s \leq 0$ there. But the integral is rescued by the exponential decay of ΔG and the fact that $0 < \kappa < \delta$ (this, essentially, is the Gap Lemma). The estimates of [GZ] show that we obtain a solution of (2.41) satisfying (2.38)(b) for $\theta < \kappa$.

Observe that if we set $D_+(z) = \det Z_+$, we have

$$\partial_z D_+ = tr(G(z) - G_{+\infty}) D_+, \tag{2.42}$$

which implies D_+ is never 0 on $[0, \infty)$.

We'll sometimes refer to $Z_\pm$ as the MZ conjugator [MZ].

2.11. **Conjugation to HP form.** In SF we need another conjugation which separates $G_{\pm\infty}$ into one $m\times m$ block whose eigenvalues vanish as $|\zeta|\to 0$ and another whose eigenvalues have real parts bounded away from zero as $|\zeta|\to 0$. Inspection of (2.37) suggests there should be such a decomposition. Indeed, we have

Lemma 2.2. *For $|\zeta|$ small there exist smooth matrices $Y_\pm(q,\zeta)$ with smooth inverses such that*

$$Y_\pm^{-1}G_{\pm\infty}Y_\pm = \begin{pmatrix} H_\pm(q,\zeta) & 0 \\ 0 & P_\pm(q,\zeta)\end{pmatrix} := G_{HP\pm}, \tag{2.43}$$

where
(2.44)

$$H_\pm = -\mathcal{A}_{\pm\infty}^{-1}\mathcal{M}_{\pm\infty} + O(|\zeta|^2) = -A_\nu(U^0_\pm, d\psi^0)^{-1}\left((i\tau+\gamma)I + g'(U^0_\pm)i\eta\right) + O(|\zeta|^2),$$
$$P_\pm = \mathcal{A}_{\pm\infty} + O(|\zeta|) = B^0(q)^{-1}A_\nu(U^0_\pm, d\psi^0) + O(|\zeta|), \text{ and}$$
$$Y_\pm(q,0) = \begin{pmatrix} I & \mathcal{A}_{\pm\infty}^{-1} \\ 0 & I\end{pmatrix}.$$

The forms of $H_\pm$ and $P_\pm$ can be motivated by a short computation which shows that for $|\zeta|$ small, small (resp. large) eigenvalues of $G_{\pm\infty}$ are close to eigenvalues of $-\mathcal{A}_{\pm\infty}^{-1}$ (resp. $\mathcal{M}_{\pm\infty}\mathcal{A}_{\pm\infty}$). One can then posit $Y_\pm$ of the given form and solve the equation

$$G_{\pm\infty}Y_\pm = Y_\pm G_{HP\pm} \tag{2.45}$$

for the entries of $Y_\pm$.

3. Lecture three: Evans functions, Lopatinski determinants, removing the translational degeneracy

Before defining the Evans function we need to make a few observations about the spectrum of $G_{\pm\infty}(q,\zeta)$.

Proposition 3.1. *1. For $\zeta\neq 0$, $\gamma\geq 0$ $G_{\pm\infty}(q,\zeta)$ each have m eigenvalues counted with multiplicities in $\Re\mu>0$ and m eigenvalues in $\Re\mu<0$.*

2. $G_{\pm\infty}(q,0)$ each have 0 as a semisimple eigenvalue of multiplicity m and nonvanishing eigenvalues equal to the eigenvalues of $\mathcal{A}_{\pm\infty}(q,0)$.

$G_{+\infty}(q,0)$ has $m-k$ eigenvalues in $\Re\mu<0$ and $G_{-\infty}(q,0)$ has $m-l$ eigenvalues in $\Re\mu>0$, where $k+l=m-1$.

Proof. **1.** Consider $G_{+\infty}(q,\zeta)$, where $\zeta\neq 0$, $\gamma\geq 0$. Then μ is an eigenvalue of $G_{+\infty}\Leftrightarrow$

$$\begin{pmatrix} 0 & I \\ \mathcal{M}_{+\infty} & \mathcal{A}_{+\infty}\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix} = \mu\begin{pmatrix} u \\ v\end{pmatrix} \Leftrightarrow v=\mu u,\ \mathcal{M}_{+\infty}u + \mathcal{A}_{+\infty}v = \mu v, \tag{3.1}$$

so $\mathcal{M}_{+\infty}u+\mathcal{A}_{+\infty}\mu u = \mu^2 u$. If $\mu = i\xi$, $\xi\in\mathbb{R}$, the latter equation is the same as
(3.2)

$$\left((i\tau+\gamma)I + A_\nu(U^0_+(q), d\psi^0)i\xi + g'(U^0_+(q))i\eta + (B^0(q)\xi^2 - 2\psi^0_y\eta\xi + \eta^2)\right)u = 0.$$

Thus, $i\tau+\gamma$ is an eigenvalue of

$$-\left(A_\nu(U^0_+(q),d\psi^0)i\xi+g'(U^0_+(q))i\eta\right)-\left(B^0(q)\xi^2-2\psi^0_y\eta\xi+\eta^2\right)$$

so hyperbolicity (H1) and positivity of the quadratic form given by the scalar second term imply

$$\gamma=-\left(B^0(q)\xi^2-2\psi^0_y\eta\xi+\eta^2\right)<0 \text{ if } (\xi,\eta)\neq 0.$$

Now $\gamma\geq 0$ so we must have $(\xi,\eta)=0$ which implies $i\tau+\gamma=0$, contradicting $\zeta\neq 0$.

We conclude that the number of eigenvalues μ with positive (or negative) real part is constant for $\zeta\neq 0$, $\gamma\geq 0$. We can then take $(\tau,\eta)=0$ and γ large to obtain an obvious count of m in each region.

2. The assertion follows immediately from (H2) and the explicit form of $G_{\pm\infty}(q,0)$. ☐

3.1. **Evans functions, instabilities, the Zumbrun-Serre result.** The Evans function is a Wronskian of solutions to

$$\partial_z U-G(q,z,\zeta)U=0 \tag{3.3}$$

which contains information about the stability of the viscous profile and, less obviously, about the stability of the original inviscid shock. For $\zeta\neq 0$ let $E_\pm(q,\zeta)$ be the set of initial data $U(0)$ such that the solution of (3.3) with that data decays to zero as $z\to\pm\infty$. As one might expect from the degeneracy (2.37), these spaces are singular in ζ near $\zeta=0$, and to resolve that singularity we blow up the origin using polar coordinates

$$\begin{aligned}&\zeta=\rho\hat\zeta,\hat\zeta=(\hat\tau,\hat\gamma,\hat\eta),\ \text{where}\\ &\hat\zeta\in S^2_+=\{\hat\zeta:|\hat\zeta|=1,\hat\gamma\geq 0\}.\end{aligned} \tag{3.4}$$

For $\zeta\neq 0$ we may just as well write $E_\pm(q,\hat\zeta,\rho)$ in place of $E^\pm(q,\zeta)$.

Proposition 3.2. *For $\zeta\neq 0$ the spaces $E_\pm(q,\hat\zeta,\rho)$ each have dimension m and are C^∞ in $(q,\hat\zeta,\rho)$. They extend continuously to $\rho=0$.*

Partial proof. Suppose $\zeta\neq 0$, and let $F_\pm(q,\zeta)$ be the analogously defined decaying spaces for

$$\partial_z V-G_{\pm\infty}(q,\zeta)V=0. \tag{3.5}$$

By Proposition 3.1 $F_\pm$ have dimension m and using the MZ conjugators we have

$$E_\pm(q,\zeta)=Z_\pm(q,0,\zeta)F_\pm(q,\zeta). \tag{3.6}$$

In particular this shows that solutions of (3.3) with initial data in $E_\pm(q,\zeta)$ decay exponentially to zero.

The continuous extension to $\rho=0$ is subtle because of glancing modes. A proof, which is best read after the reduction to *block structure* (4.24), is given in Appendix B. For now, note that when $\rho=0$ (3.5) has nonzero constant solutions of the form $(r(q,\hat\zeta),0)$. We'll see that solutions of (3.3) with data in $E_\pm(q,\hat\zeta,0)$ decay as $z\to\pm\infty$ to limits that are constant solutions of (3.5). ☐

Definition 3.1. For $\hat{\zeta} \in S^2_+$, $\rho \geq 0$ define the Evans function as the $2m \times 2m$ determinant

$$D(q,\hat{\zeta},\rho) = \det(E_+(q,\hat{\zeta},\rho), E_-(q,\hat{\zeta},\rho)). \tag{3.7}$$

Now suppose D vanishes for some (q_0,ζ_0) with $\zeta_0 = (\tau_0,\gamma_0,\eta_0)$ and $\gamma_0 > 0$. In this case we expect exponential instabilities of the boundary layers described by our approximate solution $(\tilde{u},\tilde{\psi})$. Let us explain. Vanishing of $D(q_0,\zeta_0)$ means there is a smooth solution $w(z,\zeta_0)$ of

$$L_u(q_0,z,\zeta_0,\partial_z)w = 0 \quad (L_u \text{ as in (2.23)}) \tag{3.8}$$

on the whole line $\mathbb{R}_z$ which decays exponentially to zero as $z \to \pm\infty$. Direct computation shows that

$$w^\epsilon(x,t,y) = e^{\frac{(i\tau_0+\gamma_0)t+iy\eta_0}{\epsilon}} w(\frac{x}{\epsilon},\zeta_0) \tag{3.9}$$

is then a solution of the linearized transmission problem

$$\begin{aligned} &\mathcal{L}_u(q_0,\frac{x}{\epsilon},\partial_{x,t,y})w^\epsilon = 0 \quad (\mathcal{L}_u \text{ as in (2.18)}),\\ &[w^\epsilon] = [\partial_x w^\epsilon] = 0. \end{aligned} \tag{3.10}$$

Recalling that linearized equations describe evolution of small perturbations, we see from (3.9) that some small disturbances are amplified by the factor $e^{\frac{\gamma_0 t}{\epsilon}}$, so the boundary layers described by $(\tilde{u},\tilde{\psi})$ should be completely destroyed on a time scale of $O(\epsilon)$. In this case there is no chance for L^2 estimates uniform in ϵ, and $(\tilde{u},\tilde{\psi})$ is of no help in solving the small viscosity problem.

When $\zeta = 0$ note that (3.3) is the same as the linearized profile equation

$$L_u(q,z,0,\partial_z)v = -B^0(q)\partial_z^2 v + \partial_z(A_\nu(W,d\psi^0)v) = 0. \tag{3.11}$$

Since W satisfies the profile equation (1.23)(a), it follows by differentiating that equation twice that $W_z(q,z)$ satisfies (3.11). W_z decays to zero exponentially fast as $z \to \pm\infty$, so we conclude $D(q,\hat{\zeta},0) = 0$. This degeneracy reflects the translation-invariance of profiles, so we'll sometimes refer to it as the *translational degeneracy.*

The (q,z) dependence in $G(q,z,\zeta)$ enters through the viscous profile $W(q,z)$. Our main stability hypothesis is the following *Evans hypothesis* on $W(q,z)$:

Assumption 3.1. *(H3) $D(q,\hat{\zeta},\rho)$ vanishes to exactly first order at $\rho = 0$ and has no other zeros for $(\hat{\zeta},\rho) \in S^2_+ \times \{\rho \geq 0\}$.*

The preceding discussion shows that nonvanishing of D in $\gamma > 0$ is necessary for even for linearized stability.

In the construction of the approximate solution we had to use *transversality* of the connection (recall (1.30)) and *uniform stability* of the inviscid shock. An immediate corollary of the following theorem is that Assumption 3.1 implies both of these properties.

Theorem 3.1 ([ZS]).

$$D(q,\hat{\zeta},\rho) = \beta(q)\Delta(q,\hat{\zeta})\rho + o(\rho) \ \textit{as}\ \rho \to 0, \tag{3.12}$$

where $\beta(q)$ is nonvanishing if and only if the connection is transverse at $W(q,0)$. The second factor $\Delta(q,\hat{\zeta})$ is the Majda uniform stability determinant (6.5)*, which is nonvanishing if and only if $(U^0_\pm(0,q), d\psi^0(q))$ is uniformly stable.*

A short proof is given in Appendix D.

Remark 3.1. 1. It follows from Assumption 1.3 that the stable/unstable manifolds of (1.23)(a) for the rest points $U^0_\pm(0,q)$ have dimensions $m-k$ and $m-l$ respectively, where $(m-k)+(m-l)=m+1$. Thus, the connection is transversal $\Leftrightarrow$ the intersection of these two manifolds is one dimensional $\Leftrightarrow$ the only $L^2(z)$ solution of $L_u(q,z,0,\partial_z)w=0$ on $\mathbb{R}_z$ is W_z.

2. The singularity of $\mathcal{E}'_u$ in the low frequency regime that we referred to in Lecture 2 corresponds exactly to this one dimensional kernel of L_u when $\rho=0$. The Evans hypothesis implies that for $\rho>0$, the only L^2 solution of $L_u(q,z,\zeta,\partial_z)w=0$ on $\mathbb{R}_z$ is $w=0$.

3. The main Evans hypothesis (H3) is not easy to check, so we are glad to report that in recent work by Freistühler-Szmolyan [FS] and Plaza-Zumbrun [PZ], (H3) has been shown to hold for weak Lax shocks under mild structural assumptions satisfied by some of the important physical examples.

3.2. **The Evans function as a Lopatinski determinant.** Here is an equivalent definition of the Evans function that we'll use when constructing symmetrizers. With slight abuse of notation, write the transmission problem as

$$\partial_z U_\pm - G(q,z,\zeta)U_\pm = 0 \text{ on } \pm z\geq 0,\quad \Gamma U=0 \tag{3.13}$$

where $U=(U_+,U_-)$ and $\Gamma:\mathbb{C}^{4m}\to\mathbb{C}^{2m}$ is given by $\Gamma U=U_+-U_-$.

Consider the $4m\times 4m$ determinant

$$\mathbb{D}(q,\hat{\zeta},\rho)=\det\Big(\ker\Gamma, E_+(q,\hat{\zeta},\rho)\times E_-(q,\hat{\zeta},\rho)\Big)\,. \tag{3.14}$$

Performing a few row/column operations shows that $\mathbb{D}=cD$, for some $c\in\mathbb{C}\setminus 0$. Indeed, we should expect this since, clearly, having a nontrivial intersection of the subspaces on the right side of (3.14) is equivalent to have a nontrivial intersection of $E_\pm$.

In the theory of boundary problems determinants of this sort, which measure the degree of linear independence of a subspace giving the kernel of the boundary operator with the decaying eigenspace of the interior operator, are often called *Lopatinski* determinants (ref). When these determinants are nonzero, we expect good L^2 estimates; when they vanish, we expect degenerate estimates or no estimates. Indeed, we saw above that vanishing of D in $\gamma>0$ leads to exponential blowup. The vanishing of D at $\rho=0$ (which implies $\gamma=\rho\hat{\gamma}=0$) because of the translational degeneracy is a borderline case, and leads to degenerate L^2 estimates [GMWZ1, GMWZ2].

3.3. **Doubling.** For future use and to make the connection to boundary problems more explicit, let's double the problem (3.13) and write it as a $4m\times 4m$ system on $z\geq 0$. If $f(z)$ is any function (complex valued, matrix valued,...) defined on $z\leq 0$, we set

$$\tilde{f}(z)=f(-z) \text{ for } z\geq 0. \tag{3.15}$$

Then the $2m \times 2m$ transmission problem on $\mathbb{R}_z$

$$\partial_z U_\pm - G(q,z,\zeta)U_\pm = F_\pm,\;\; U_+ - U_- = 0 \tag{3.16}$$

is equivalent to the $4m \times 4m$ boundary problem on $z \geq 0$

$$\partial_z U - \mathcal{G}(q,z,\zeta)U = \mathcal{F},\;\; \Gamma U = 0, \tag{3.17}$$

where $U = (U_+, \tilde{U}_-)$, $\mathcal{F} = (F_+, -\tilde{F}_-)$, $\Gamma U = U_+ - \tilde{U}_-$, and

$$\mathcal{G} = \begin{pmatrix} G & 0 \\ 0 & -\tilde{G} \end{pmatrix}. \tag{3.18}$$

Note that

$$\mathbb{E}(q,\hat{\zeta},\rho) := E_+(q,\hat{\zeta},\rho) \times E_-(q,\hat{\zeta},\rho) \tag{3.19}$$

is the decaying generalized eigenspace for $U_z - \mathcal{G}U = 0$ on $z \geq 0$ and $\mathbb{D}$ is now exactly the Lopatinski determinant (in the classical sense) for the boundary problem (3.17).

It's easy to check that if we use the MZ conjugators $Z_\pm$ to define

$$Z(q,z,\zeta) = \begin{pmatrix} Z_+ & 0 \\ 0 & \tilde{Z}_- \end{pmatrix} \tag{3.20}$$

and define V by $U = ZV$, then U satisfies (3.17) $\Leftrightarrow$ V satisfies

$$\partial_z V - \mathcal{G}_\infty V = Z^{-1}\mathcal{F},\;\; \Gamma Z V = 0, \tag{3.21}$$

where

$$\mathcal{G}_\infty = \begin{pmatrix} G_{+\infty} & 0 \\ 0 & -G_{-\infty} \end{pmatrix}. \tag{3.22}$$

One advantage of doubling is that the two distinct limiting problems have become a single limiting problem at $z = +\infty$. Observe that each additional conjugation twists the boundary condition by an additional matrix factor to the right of Γ. We won't use the doubled form until the symmetrizer construction in Lecture 4.

3.4. **Slow modes and fast modes.** Recall that the problems on $\pm z \geq 0$

$$U_z - GU = 0, V_z - G_{\pm\infty}V = 0, W_z - G_{HP\pm}W = 0 \tag{3.23}$$

are related by the conjugators: $U = ZV$, $V = YW$.

We have already defined the decaying spaces $E_\pm$, $F_\pm$ for the first two problems. For $\rho \neq 0$ set $K_\pm(q,\zeta)$ equal to the decaying generalized eigenspaces for

$$\partial_z W - G_{HP\pm}W = 0. \tag{3.24}$$

Clearly, $F_\pm = Y_\pm K_\pm$ and $E_\pm = (Z_\pm|_{z=0})Y_\pm K_\pm$.

Write $H_\pm(q,\hat{\zeta},\rho) = \rho \hat{H}_\pm(q,\hat{\zeta},\rho)$ where

$$\hat{H}(q,\hat{\zeta},0) = -A_\nu(U^0_\pm, d\psi^0)^{-1}\left((i\hat{\tau} + \hat{\gamma})I + g'(U^0_\pm)i\hat{\eta}\right). \tag{3.25}$$

Using obvious notation we can in turn decompose $K_\pm$ as

$$K_\pm(q,\hat{\zeta},\rho) = K_{\hat{H}_\pm}(q,\hat{\zeta},\rho) \oplus K_{P_\pm}(q,\zeta). \tag{3.26}$$

By (H1) the dimensions of $K_{\hat{H}_\pm}$ are k (resp. l), while the dimensions of $K_{P_\pm}$ are $m-k$ (resp. $m-l$). By the result proved in Appendix B, $K_\pm(q,\hat{\zeta},\rho)$ extend continuously to $\rho=0$. The main step is to prove continuous extendability of $K_{\hat{H}_\pm}$.

We may now define *slow* (resp. *fast*) *modes* as solutions of (3.3) of the form

$$U_\pm = Z_\pm Y_\pm W_\pm, \tag{3.27}$$

where $W_\pm(q,z,\hat{\zeta},\rho)$ is a solution of (3.24) with $W_\pm(q,0,\hat{\zeta},\rho)\in K_{\hat{H}_\pm}$ (resp. $K_{P_\pm}$).

Fast modes decay exponentially to zero as $z\to\pm\infty$ for $\rho\geq 0$, and slow modes decay exponentially to zero (but slowly) for $\rho>0$ small. For $\rho=0$ slow modes can decay to nonzero constant vectors. In Appendix C we identify those limits, a knowledge of which is needed for the Zumbrun-Serre result (Appendix D) and also for removal of the translational degeneracy. Here is the result on slow modes.

Proposition 3.3. *Let $U_\pm(q,z,\hat{\zeta},\rho)$ be a slow mode. Then*

$$\lim_{z\to\pm\infty} U_\pm(q,z,\hat{\zeta},0) \text{ exists and belongs to } K_{\hat{H}_\pm}(q,\hat{\zeta},0). \tag{3.28}$$

Let's rephrase this in a manner useful for the applications. The spaces $K_{\hat{H}_\pm}(q,\hat{\zeta},\rho)$ have bases

$$\{(r_+^s(q,\hat{\zeta},\rho),0)\}_{s=1}^k \text{ and } \{(r_-^t(q,\hat{\zeta},\rho),0)\}_{t=1}^l \tag{3.29}$$

respectively, where the $r_\pm^j$ are smooth in $\rho>0$ and extend continuously to $\rho=0$. Since

$$\hat{H}_\pm(q,\hat{\zeta},0)=\mathbb{H}_\pm(q,\hat{\zeta}) \text{ (for } \mathbb{H} \text{ as in Appendix A)}, \tag{3.30}$$

we may choose the $r_\pm^j$ to agree at $\rho=0$ with the vectors $r_\pm^j(q,\hat{\zeta})$ that appear in the definition of the uniform stability determinant $\Delta(q,\hat{\zeta})$ (6.5). Thus, for each choice of sign

$$\lim_{z\to\pm\infty} U_\pm(q,z,\hat{\zeta},0)\in \operatorname{span}\{(r_\pm^j(q,\hat{\zeta}),0)\}, \tag{3.31}$$

where $r_\pm^j(q,\hat{\zeta})$ span the decaying generalized eigenspaces of $\mathbb{H}_\pm$ as in Appendix A.

3.5. **Removing the translational degeneracy.** Our next main task is to show how the extra boundary condition can be used to remove the translational degeneracy. In lecture two we sketched a strategy for reducing study of the fully linearized problem to the partially linearized one ($\mathcal{E}_u'$). Let's recall that here in the context of our ODE problem.

Returning to transmission notation, where $U=(v,v_z)$, we recall (2.30) the fully linearized transmission problem

$$\begin{aligned}&\partial_z U - G(q,z,\zeta)U = F + \begin{pmatrix}0\\ -(B^0(q))^{-1}L_\psi(q,z,\zeta)\phi)\end{pmatrix} = F+\mathcal{B}(q,z,\zeta)\phi\\ &[U]=0,\ p(\zeta)\phi - l(q)\cdot v = 0 \text{ on } z=0,\end{aligned} \tag{3.32}$$

where $p(\zeta)=(i\tau+\gamma+\eta^2)$.

Set $P(q,z)=(W_z,W_{zz})$. The relationship (2.28) between the partial linearizations

$$L_u(q,z,\zeta,\partial_z)W_z = -L_\psi(q,z,\zeta) \tag{3.33}$$

is equivalently expressed as

$$(\partial_z - G)P = \mathcal{B}. \tag{3.34}$$

Thus, (U,ϕ) satisfies (3.32) $\Leftrightarrow$ $(v^\sharp, v_z^\sharp) = U^\sharp := U - P\phi$ satisfies

$$\partial_z U^\sharp - G(q,z,\zeta)U^\sharp = F, \quad [U^\sharp]=0, \tag{3.35}$$

together with the extra boundary condition

$$l(q)\cdot v^\sharp = (p(\zeta)+1)\phi = (1+i\tau+\gamma+\eta^2)\phi \text{ on } z=0. \tag{3.36}$$

Now as we've seen, the main Evans hypothesis implies the Lopatinski determinant $\mathbb{D}(q,\hat\zeta,\rho)$ for the problem (3.35) is nonvanishing in MF or HF, so we expect good estimates there, including an estimate for the trace $|v^\sharp(0)|$. The extra condition (3.36) will then allow us to estimate $|\phi|$ in terms of $|v^\sharp(0)|$.

On the other hand we can't use this approach to get estimates uniform with respect to ζ in SF since in the notation of (3.14)

(3.37)

$$\ker\Gamma \cap (E_+(q,\hat\zeta,0)\times E_-(q,\hat\zeta,0)) = \mathrm{span}(W_z(q,0),W_{zz}(q,0),W_z(q,0),W_{zz}(q,0)),$$

so (3.20) is singular in SF.

To handle SF, instead of $U^\sharp$ as in (3.35) we define as our "good unknown"

$$U^\flat = U - R(q,z,\zeta)\phi, \tag{3.38}$$

where $R_\pm = (r_\pm, s_\pm)$ is smooth in $\pm z\geq 0$ and constructed to satisfy

$$\begin{aligned}&(a)\partial_z R - G(q,z,\zeta)R = \mathcal{B} \text{ in } \pm z\geq 0, \quad l(q)\cdot r_\pm(q,0,\zeta) = p(\zeta)\\ &(b)R(q,z,0)=0, \quad |R(q,z,\zeta)|\leq Ce^{-\delta|z|},\end{aligned} \tag{3.39}$$

where G, $\mathcal{B}$, and p are as in (3.32).

The construction is not hard and is given in Appendix E. For now we note that since $\mathcal{B}(q,z,\zeta)$ also satisfies (3.39)(b) and we can use MZ conjugators to reduce to a constant coefficient problem, we should expect such R to exist.

Since $R_\pm(q,z,0)=0$ we may write

$$R_\pm(q,z,\zeta) = \rho\hat R_\pm(q,z,\hat\zeta,\rho). \tag{3.40}$$

Now (U,ϕ) satisfies (3.32)$\Leftrightarrow$ $U^\flat = (u^\flat, u_z^\flat)$ satisfies

$$\begin{aligned}&(a)\partial_z U^\flat - G(q,z,\zeta)U^\flat = F,\\ &(b)[U^\flat] = -\phi[R] = -\phi\rho[\hat R] = -\hat\phi[\hat R],\\ &(c)l(q)\cdot u_+^\flat = l\cdot v - \phi l\cdot r_+ = p(\zeta)\phi - p(\zeta)\phi = 0.\end{aligned} \tag{3.41}$$

The following fact, proved below, is a consequence of uniform stability of the inviscid shock.

Proposition 3.4. *For all $\hat\zeta\in S_+^2$, we have $[\hat R(q,0,\hat\zeta,0]\neq 0$.*

So for ρ small there is a well-defined orthogonal projection

$$\pi(q,\hat{\zeta},\rho):\mathbb{C}^{2m}\to\mathbb{C}^{2m} \tag{3.42}$$

onto $[\hat{R}(q,0,\hat{\zeta},\rho)]^{\perp}$, the subspace of $\mathbb{C}^{2m}$ orthogonal to $[\hat{R}(q,0,\hat{\zeta},\rho)]$.

Now we can apply $\pi(q,\hat{\zeta},\rho)$ to (3.41)(b) to project out the front, giving us the following problem for $U^\flat$:

$$\begin{aligned}&\partial_z U^\flat - G(q,z,\zeta)U^\flat = F\\ &\pi(q,\hat{\zeta},\rho)[U^\flat]=0,\ l(q)\cdot u_+^\flat = 0.\end{aligned} \tag{3.43}$$

Using a notation like the one we used in defining $\mathbb{D}(q,\hat{\zeta},\rho)$, let

$$\tilde{\Gamma}:\mathbb{C}^{4m}\to\mathbb{C}^{2m+1} \tag{3.44}$$

denote the boundary condition in (3.43). It's easy to check that $\dim(\ker\tilde{\Gamma})=2m$.

The next Proposition, a key to the whole analysis, expresses the fact that the modified problem (3.43) *does* satisfy the Lopatinski condition in SF:

Proposition 3.5. *Assume (H1), (H2), and (H3). Then for all* $\hat{\zeta}\in S^2_+$

$$\tilde{D}(q,\hat{\zeta},0)=\det\Big(\ker\tilde{\Gamma},E_+(q,\hat{\zeta},0)\times E_-(q,\hat{\zeta},0)\Big)\neq 0,$$

or equivalently,

$$\ker\tilde{\Gamma}\cap(E_+(q,\hat{\zeta},0)\times E_-(q,\hat{\zeta},0))=\{0\}.$$

By continuity we have $\tilde{D}(q,\hat{\zeta},\rho)\neq 0$ for ρ small. In other words the translational degeneracy has been removed. In view of our earlier discussion of Lopatinski determinants, we may expect good L^2 estimates for (3.43), including an estimate for the trace $U^\flat(0)$. Since $[U^\flat]=-\hat{\phi}[\hat{R}]$ and $[\hat{R}]$ is nonvanishing for ρ small (Proposition 3.4), we can then estimate $|\hat{\phi}|$ in terms of $|U^\flat(0)|$.

Remark 3.2. The proof of Proposition 3.5, given in Appendix E, is based on nonvanishing of the uniform stability determinant $\Delta(q,\hat{\zeta})$. It is easy to see that if $(U_+(0),U_-(0))$ is a nonvanishing element of

$$E_+(q,\hat{\zeta},0)\times E_-(q,\hat{\zeta},0),$$

the boundary condition $l(q)\cdot u_+=0$ implies

$$(U_+(0),U_-(0))\notin\ker\Gamma \tag{3.45}$$

(use (3.37) and the fact that $l(q)\cdot W_z(q,0)=-1$). However, to prove the Proposition we need to show that $(U_+(0),U_-(0))$ is not in the larger space $\ker(\pi(q,\hat{\zeta},0)\Gamma)$.

4. Lecture four: Block structure, symmetrizers, estimates

In the construction of symmetrizers it is most convenient to work with the $4m\times 4m$ doubled problem (3.17)

$$\partial_z U-\mathcal{G}(q,z,\zeta)U=\mathcal{F},\ \Gamma U=0, \tag{4.1}$$

where $U=(U_+,\tilde U_-)$, $\Gamma U=U_+-\tilde U_-$, and

$$\mathcal{G}=\begin{pmatrix}G & 0\\ 0 & -\tilde G\end{pmatrix}. \tag{4.2}$$

This is because the problems in $\pm z\geq 0$ are coupled by the boundary condition which depends on both U_+ and U_-, and the boundary estimates can't be done for $\pm$ blocks separately. Symmetrizer construction is much easier after further conjugation of $\mathcal{G}$ to an appropriate block structure, which is different in each of the three frequency regimes.

4.1. **The MF regime.** We've already described the MZ conjugation to the doubled limiting problem (3.21) on $z\geq 0$

$$\begin{aligned}&\partial_z V-\mathcal{G}_\infty V=F,\ \Gamma ZV=0,\ \text{where}\\ &\mathcal{G}_\infty=\begin{pmatrix}G_{+\infty} & 0\\ 0 & -G_{-\infty}\end{pmatrix}.\end{aligned} \tag{4.3}$$

This conjugation can't be done uniformly in HF, so we use it only in the SF and MF regimes. Here we'll carry out the full details in MF, the easiest of the three regimes. This will clarify how Lopatinski conditions yield estimates.

We work near a basepoint $(\underline{q},\underline{\zeta})$ with $\underline{\zeta}\neq 0$. Proposition 3.1 implies that for (q,ζ) near $(\underline{q},\underline{\zeta})$ the spectrum of $\mathcal{G}_\infty$ lies in a compact subset of the complement of the imaginary axis, with $2m$ eigenvalues (counted with multiplicity) on each side. Thus, there exists a smooth conjugator $T_M(q,\zeta)$ such that

$$T_M^{-1}\mathcal{G}_\infty T_M=\begin{pmatrix}P_g(q,\zeta) & 0\\ 0 & P_d(q,\zeta)\end{pmatrix}:=\mathcal{G}_{gd} \tag{4.4}$$

where for some $C>0$

$$\Re P_g:=\frac{P_g+P_g^*}{2}>CI;\ \Re P_d<-CI. \tag{4.5}$$

Note that each block in (4.4) is of size $2m$ with, for example, m of the eigenvalues of P_g coming from $G_{+\infty}$ and m from $-G_{-\infty}$. The letters g and d stand for "growing" and "decaying" respectively.

Redefining F and setting $V=T_MU$ (so $U=(U^1,U^2)$ is not the same as before), we have reduced the study of (4.3) to the study of

$$\begin{aligned}&\partial_z U-\mathcal{G}_{gd}U=F\text{ in }z\geq 0\\ &\Gamma'(q,\zeta)U:=\Gamma Z(q,0,\zeta)T_M(q,\zeta)U=0\text{ on }z=0.\end{aligned} \tag{4.6}$$

In Lecture 3 we defined the decaying generalized eigenspaces $E_\pm(q,\zeta)$ for $G(q,z,\zeta)$ on $\pm z\geq 0$. Observe that

$$\mathbb{E}(q,\zeta):=E_+(q,\zeta)\times E_-(q,\zeta) \tag{4.7}$$

is the $2m$ dimensional decaying space for $\mathcal{G}$. The decaying space for $\mathcal{G}_{gd}(q,\zeta)$

$$\mathbb{F}(q,\zeta):=\{(0,U^2):U^2\in\mathbb{C}^{2m}\} \tag{4.8}$$

is independent of (q,ζ) and evidently satisfies $\mathbb{E}(q,\zeta)=Z(q,0,\zeta)T_M(q,\zeta)\mathbb{F}$.

The main Evans hypothesis (H3) implies that for (q,ζ) near the basepoint

$$\begin{aligned} &\ker \Gamma \cap \mathbb{E}(q,\zeta) = \{0\}, \text{ or equivalently} \\ &\ker \Gamma'(q,\zeta) \cap \mathbb{F} = \{0\}. \end{aligned} \tag{4.9}$$

We're ready now to construct the symmetrizer. In fact, with the above preparation the construction is practically trivial. We'll write

$$U = U_g + U_d, \text{ where } U_g = (U^1, 0) \text{ and } U_d = (0, U^2). \tag{4.10}$$

We'll use the notation

$$|U|_2 = |U(z)|_{L^2(z)}, \ |U| = |U(0)|_{\mathbb{C}^{4m}}, \ (U,V) = \int_0^\infty \langle U(z), V(z)\rangle dz, \tag{4.11}$$

where $\langle \ , \ \rangle$ is the inner product in $\mathbb{C}^{4m}$.

The symmetrizer for (4.6) in MF is a matrix $S(q,z,\zeta)$ with the following properties

$$\begin{aligned} &(a) S = S^*, \ |S(q,z,\zeta)| \leq C \\ &(b) \Re(S\mathcal{G}_{gd}) \geq I \text{ in } z \geq 0 \\ &(c) S + C(\Gamma')^*\Gamma' \geq I \text{ on } z = 0 \end{aligned} \tag{4.12}$$

for some $C > 0$. We may take S to be simply

$$\begin{pmatrix} cI & 0 \\ 0 & -I \end{pmatrix} \tag{4.13}$$

for some large enough $c > 0$ to be chosen. Clearly, properties (4.12)(a),(b) are then satisfied.

The Lopatinski condition (4.9) is then equivalent to

$$|\Gamma'(q,\zeta)U_d| \geq C|U_d| \text{ for some } C > 0 \tag{4.14}$$

near the basepoint, and this implies

$$|U_d|^2 \leq C|\Gamma' U_d|^2 \leq C_1(|\Gamma' U|^2 + |U_g|^2). \tag{4.15}$$

To arrange (4.12)(c) observe that on $z = 0$

$$\begin{aligned} \langle SU, U\rangle &= c|U_g|^2 - |U_d|^2 = c|U_g|^2 + |U_d|^2 - 2|U_d|^2 \\ &\geq c|U_g|^2 + |U_d|^2 - 2C_1(|\Gamma' U|^2 + |U_g|^2) \\ &= (c - 2C_1)|U_g|^2 + |U_d|^2 - 2C_1|\Gamma' U|^2, \end{aligned} \tag{4.16}$$

which gives (4.12)(c) for c big enough.

Integration by parts, the equation, and the property $S = S^*$ yield the identity

$$-\langle SU(0), U(0)\rangle = \int_0^\infty \partial_z \langle SU, U\rangle dz = 2\Re(S\mathcal{G}_{gd}U, U) + 2\Re(SF, U), \tag{4.17}$$

which together with (4.12) easily implies the L^2 estimate

$$|U|_2^2 + |U|^2 \leq C(|F|_2^2 + |\Gamma' U|^2), \tag{4.18}$$

for a C independent of (q,ζ) near the basepoint. Here we've used

$$|(SF, U)| \leq \delta |U|_2^2 + C_\delta |F|_2^2 \tag{4.19}$$

and absorbed the $\delta|U|_2^2$ from the right. Conjugation via ZT_M and compactness of MF then yield the uniform estimate for (4.1) in MF

$$|U|_2^2 + |U|^2 \leq C(|F|_2^2 + |\Gamma U|^2). \tag{4.20}$$

Recall that the q dependence enters only through $W(q,z)$.

4.2. **The SF regime.** Symmetrizer construction was rather easy in MF thanks mainly to three things: the MZ conjugator, the compactness of MF, and the fact that for ζ in MF the spectrum of $\mathcal{G}_\infty$ is contained in $K_+ \cup K_-$ for compact sets $K_\pm$ in $\pm\Re\mu > 0$ respectively.

SF is more subtle because, even though we have the MZ conjugator and compactness here, the third property does not hold. As $\rho \to 0$ the spectrum creeps right up to the imaginary axis $\Re\mu = 0$ from both sides (and in a rather singular way). Indeed, recall that at $\zeta = 0$ we have

$$G_{\pm\infty}(q,0) = \begin{pmatrix} 0 & I \\ 0 & \mathcal{A}_{\pm\infty}(q,0) \end{pmatrix}. \tag{4.21}$$

In SF we first conjugate (4.1) via Z to $\mathcal{G}_\infty$, then again via

$$Y(q,\zeta) = \begin{pmatrix} Y_+ & 0 \\ 0 & Y_- \end{pmatrix} \text{ to}$$

$$\tilde{\mathcal{G}}_{HP}(q,\zeta) = \begin{pmatrix} H_+ & 0 & 0 & 0 \\ 0 & P_+ & 0 & 0 \\ 0 & 0 & -H_- & 0 \\ 0 & 0 & 0 & -P_- \end{pmatrix} \tag{4.22}$$

(recall (2.43)), and next via a constant matrix T_c to

$$\mathcal{G}_{HP}(q,\zeta) = \begin{pmatrix} H_+ & 0 & 0 & 0 \\ 0 & -H_- & 0 & 0 \\ 0 & 0 & P_+ & 0 \\ 0 & 0 & 0 & -P_- \end{pmatrix}. \tag{4.23}$$

We may write $H_\pm(q,\zeta) = \rho\hat{H}_\pm(q,\hat{\zeta},\rho)$.

So far the conjugations work uniformly for ρ small. For the final conjugation to *block structure* $\mathcal{G}_B$, we fix a basepoint $(\underline{q},\underline{\hat{\zeta}},0)$ and construct a matrix $T_B(q,\hat{\zeta},\rho)$ such that:

$$T_B^{-1}\mathcal{G}_{HP}T_B = \begin{pmatrix} H_B(q,\hat{\zeta},\rho) & 0 & 0 \\ 0 & P_g(q,\zeta) & 0 \\ 0 & 0 & P_d(q,\zeta) \end{pmatrix} := \mathcal{G}_B(q,\hat{\zeta},\rho) \tag{4.24}$$

for $(q,\hat{\zeta},\rho)$ near $(\underline{q},\underline{\hat{\zeta}},0)$. Here T_B is of the form

$$T_B = \begin{pmatrix} T_{BH}(q,\hat{\zeta},\rho) & 0 \\ 0 & T_{BP}(q,\zeta) \end{pmatrix} \tag{4.25}$$

with blocks of size $2m$, while the blocks H_B, P_g, and P_d are of sizes $2m$, $m-1$, and $m+1$ respectively. All the blocks appearing in T_B and $\mathcal{G}_B$ are smooth functions of their arguments.

The eigenvalues of P_g and P_d lie in compact subsets $K_\pm$ of $\pm\Re\mu > 0$ respectively. Their dimensions can be read off from Proposition 3.1. In addition

$$\Re P_g = \frac{1}{2}(P_g + P_g^*) \geq cI \text{ and } \Re P_d \leq -cI \tag{4.26}$$

for some $c > 0$.

The eigenvalues of H_B are those of $\mathcal{G}_\infty$ that approach zero as $\rho \to 0$. We may write $H_B(q, \hat\zeta, \rho) = \rho \hat H_B(q, \hat\zeta, \rho)$ where $\hat H$ has the block structure described in the following lemma. Observe that the matrix $\hat H_B$ is a perturbation, depending on an additional parameter ρ, of the matrix that arose in the hyperbolic analysis of Kreiss [K].

Lemma 4.1. *The conjugating block T_{BH} may be constructed so that*

$$\hat H_B(q, \hat\zeta, \rho) = \begin{bmatrix} Q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_s \end{bmatrix}, \tag{4.27}$$

where the blocks Q_k are $\nu_k \times \nu_k$ matrices which satisfy one of the following conditions for $(q, \hat\zeta, \rho)$ near $(\underline{q}, \underline{\hat\zeta}, 0)$:

i) $\Re Q_k > 0$.

ii) $\Re Q_k < 0$.

iii) $\nu_k = 1$, $\Re Q_k = 0$ when $\hat\gamma = \rho = 0$, and $\partial_{\hat\gamma}(\Re Q_k)\partial_\rho(\Re Q_k) > 0$.

iv) $\nu_k > 1$, $Q_k(q, \hat\zeta, \rho)$ has purely imaginary coefficients when $\hat\gamma = \rho = 0$, there is $\alpha_k \in \mathbb{R}$ such that

$$Q_k(\underline{q}, \underline{\hat\zeta}, 0) = i \begin{bmatrix} \alpha_k & 1 & 0 & \\ 0 & \alpha_k & \ddots & 0 \\ & \ddots & \ddots & 1 \\ & & \cdots & \alpha_k \end{bmatrix}, \tag{4.28}$$

and the lower left corner a of Q_k satisfies $\partial_{\hat\gamma}(\Re a)\partial_\rho(\Re a) > 0$.

Sketch of the construction of T_B. **1.** Starting from $\mathcal{G}_{HP}$ we construct smooth projectors $\mathcal{P}_g(q, \zeta)$ and $\mathcal{P}_d(q, \zeta)$

$$\mathcal{P}_{g,d}(q, \zeta) = \frac{1}{2\pi i} \int_{\mathcal{C}_{g,d}} (\xi - \begin{pmatrix} P_+ & 0 \\ 0 & -P_- \end{pmatrix})^{-1} d\xi \tag{4.29}$$

using contours $\mathcal{C}_g$ and $\mathcal{C}_d$ in the right and left half planes, respectively, which enclose the eigenvalues of the submatrix $(P_+, -P_-)$ in those half planes. Applying $\mathcal{P}_{g,d}(q, \zeta)$ to a basis of $\mathcal{P}_{g,d}(q, 0)$ yields a basis of

$$\text{range } \mathcal{P}_{g,d}(q, \zeta)$$

varying smoothly with ζ.

The blocks H_+ and $-H_-$ are conjugated separately to block structure. Thus, there is a k_0 such that the blocks $Q_1, \ldots, Q_{k_0}$ in $\hat H_B$ correspond to $\hat H_+$, while blocks $Q_{k_0+1}, \ldots, Q_p$ correspond to $-\hat H_-$. For example, suppose

$$\{i\alpha_1, \ldots, i\alpha_{k_0-2}\} \tag{4.30}$$

are the distinct pure imaginary eigenvalues of $\hat{H}_+(\underline{q},\underline{\hat{\zeta}},0)$ with $i\alpha_j$ of multiplicity m_j, and let $m_\pm$ be the number of eigenvalues of $\hat{H}_+(\underline{q},\underline{\hat{\zeta}},0)$ counted with multiplicity in $\pm\Re\mu > 0$. Using projectors defined by integration on suitable contours as above, one obtains a decomposition of $\hat{H}_+$ near the basepoint in which there is a block of size m_j satisfying either (iii) or (iv) corresponding to each $i\alpha_j$, a block of size m_+ satisfying (i), and a block of size m_- satisfying (ii). As the basepoint changes so does the decomposition.

A further change of basis in cases (iii),(iv) puts $Q_k(\underline{q},\underline{\hat{\zeta}},0)$ in Jordan form. Changing basis again using Ralston's Lemma [Ra, CP] makes Q_k pure imaginary in cases (iii),(iv) when $\hat{\gamma} = \rho = 0$. Observe that by hyperbolicity blocks satisfying conditions (iii) or (iv) only arise when $\underline{\hat{\gamma}} = 0$.

2. We discuss the crucial *sign condition* in cases (iii) and (iv) in the next subsection. □

4.3. **The sign condition.** Suppose we start with $\hat{H}_+(\underline{q},\underline{\hat{\zeta}},0)$, where we assume $\underline{\hat{\gamma}} = 0$ as in cases (iii) and (iv) above, and then perturb to $\hat{\gamma} > 0$ while holding $\rho = 0$ (case A), or to $\rho > 0$ holding $\underline{\hat{\gamma}} = 0$ (case B). In both cases the perturbed matrix $\hat{H}_+$ has $m - k$ eigenvalues with positive real part and k with negative real part.

In case A this follows from hyperbolicity (H1), (H2), and the explicit form of $\hat{H}_+$ (2.44). Hyperbolicity implies that the number of eigenvalues with positive (or negative) real part is constant for $\hat{\gamma} > 0$, $\rho = 0$. We can then take $(\hat{\tau}, \hat{\eta}) = 0$ to obtain a count. In case B the count follows directly from (H2) and Proposition (3.1).

To describe this situation we'll say that changes in $\hat{\gamma}$ or ρ lead to *identical splitting* for $\hat{H}_+(\underline{q},\underline{\hat{\zeta}},0)$ (this should not be confused with "consistent splitting", which has a different meaning).

Identical splitting follows from the sign condition in (iii) and (iv), but does not quite imply it. The sign condition implies more; namely, that we have identical splitting for each block Q_k. This is because the behavior of the lower left entry governs the splitting of $i\alpha_k$ when a block $Q_k(\underline{q},\underline{\hat{\zeta}},0)$ has the Jordan form (4.28). The sign condition implies changes in $\hat{\gamma}$ or ρ have the same effect.

Let's try to understand the role of the lower left entry in a simple case. For $\alpha \in \mathbb{R}$ set

$$Q(\gamma) = i\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix} + \gamma(b_{ij}) \tag{4.31}$$

where (b_{ij}) is some 3×3 constant matrix with $b_{31} \neq 0$. Compute the characteristic polynomial of $Q(\gamma)$ by expanding down the first column to get

$$\det(Q(\gamma) - \xi I) = (i\alpha - \xi)^3 - \gamma b_{31} + O(\gamma^2) + O(\gamma(i\alpha - \xi)). \tag{4.32}$$

Eigenvalues $\xi(\gamma)$ satisfy

$$i\alpha - \xi(\gamma) = o(1) \text{ as } \gamma \to 0, \tag{4.33}$$

so to determine the splitting we can solve

$$(i\alpha - \xi)^3 = \gamma b_{31} \tag{4.34}$$

for ξ.

When $\Re b_{31} > 0$ we obtain two solutions with $\Re\xi > 0$, one with $\Re\xi < 0$. When $\Re b_{31} < 0$ we obtain one solution with $\Re\xi > 0$, two with $\Re\xi < 0$.

The sign condition allows one to construct symmetrizers by a modification of the ansatz used by Kreiss in [K]. In [K] there was only one perturbation parameter $\hat{\gamma}$ and one derivative to consider $\partial_{\hat{\gamma}}\Re a$. That derivative was nonzero as a consequence of his strict hyperbolicity assumption. Because of our sign condition, we can construct a symmetrizer in this two-parameter situation by adding an extra term, corresponding to the ρ parameter, to the kth block of S in the Kreiss ansatz. As explained in Appendix B the sign condition is also used in the proof that the decaying eigenspaces $E_\pm(q,\hat{\zeta},\rho)$ extend continuously to $\rho = 0$.

4.4. **The SF estimate.** Using the conjugator ZYT_cT_B we have reduced the study of

$$\begin{aligned} &U_z - \mathcal{G}(q,z,\zeta)U = F,\ \tilde{\Gamma}U = 0 \text{ to} \\ &U_z - \mathcal{G}_B(q,\hat{\zeta},\rho)U = F,\ \tilde{\Gamma}ZYT_cT_BU := \Gamma' U = 0. \end{aligned} \tag{4.35}$$

Here $\tilde{\Gamma}$ is the Lopatinski boundary condition from (3.44) obtained by removing the translational degeneracy.

The block form (4.24) of $\mathcal{G}_B$ determines a partition of $U \in \mathbb{C}^{4m}$ as $U = (u_1,\dots,u_s,u_+,u_-)$, where $u_\pm$ correspond to the blocks P_g, P_d respectively. We caution that in the remainder of this subsection the meaning of $\pm$ is completely different from earlier usage.

Denote by β_j the number of eigenvalues of Q_j with $\Re\mu < 0$ for $\hat{\gamma} > 0$ (or $\rho > 0$), and write

$$u_j = (u_{j-}, u_{j+}) \tag{4.36}$$

where u_{j-} consists of the first β_j components of u_j.

Next set

$$\begin{aligned} U_{P+} &= (0,\dots,0,u_+,0) \\ U_{P-} &= (0,\dots,0,0,u_-) \\ U_{H+} &= ((0,u_{1+}),\dots,(0,u_{s+}),0,0) \\ U_{H-} &= ((u_{1-},0),\dots,(u_{s-},0),0,0), \end{aligned} \tag{4.37}$$

and write

$$\begin{aligned} U &= U_{P_+} + U_{P_-} + U_{H_+} + U_{H_-} \\ U_\pm &= U_{P_\pm} + U_{H_\pm} \\ U_P &= U_{P_+} + U_{P_-} \\ U_H &= U_{H_+} + U_{H_-}. \end{aligned} \tag{4.38}$$

The definition of U_{H-} is based on the observation, proved in Appendix B, that $\{(u_{j-},0) : u_{j-} \in \mathbb{C}^{\beta_j}\}$ is precisely the continuous extension *at the basepoint* of the decaying space corresponding to the block Q_j. Note that the analogous statement is *not* true for U_{H_+}, because of the fact that continuous extensions of growing and decaying spaces have nontrivial intersection (and sometimes even coincide) at glancing basepoints. Rather, U_{H_+} is just a convenient choice of complementary vector.

The symmetrizer S has the form

$$S(q,\hat\zeta,\rho)=\begin{pmatrix} S_1(q,\hat\zeta,\rho) & & & \\ & \ddots & & \\ & & S_s(q,\hat\zeta,\rho) & \\ & & & S_P(q,\zeta)\end{pmatrix}, \tag{4.39}$$

where the S_j, S_P are C^∞ functions of their arguments. We'll sometimes write

$$S=\begin{pmatrix} S_H & \\ & S_P\end{pmatrix}, \tag{4.40}$$

where each block is of size $2m$.

Let

$$U_{H_j}=U_{H_{j+}}+U_{H_{j-}} \tag{4.41}$$

where the terms on the right have obvious meanings in view of (4.37). The S_j are constructed so that $S=S^*$, with interior estimates

$$\begin{aligned}&(\text{Re } S\mathcal{G}_B U_P,U_P)\geq C|U_P|_2^2\\ &(\text{Re } S\mathcal{G}_B U_{H_j},U_{H_j})\geq(\gamma+\rho^2)|U_{H_j}|_2^2,\end{aligned} \tag{4.42}$$

as well as boundary estimates

$$\begin{aligned}&(a)\ (SU_P,U_P)\geq c|U_{P_+}|^2-|U_{P_-}|^2\\ &(b)\ (SU_{H_j},U_{H_j})\geq c|U_{H_{j+}}|^2-|U_{H_{j-}}|^2\end{aligned} \tag{4.43}$$

both holding uniformly near the basepoint.

Note that S_P can be taken to be simply

$$S_P=\begin{pmatrix} cI & \\ & -I\end{pmatrix} \tag{4.44}$$

for some large $c>0$, where the blocks are of sizes $m-1$ and $m+1$. Details of the construction of S_H are the same as for the case of Dirichlet boundary layers and are given in [MZ].

Since Γ' satisfies the uniform Lopatinski condition at $(\underline{q},\underline{\hat\zeta},0)$ we have

$$|U_-|^2\leq C|\Gamma' U_-|^2\leq C(|\Gamma' U|^2+|U_+|^2) \tag{4.45}$$

at $(\underline{q},\underline{\hat\zeta},0)$ and in fact uniformly near the basepoint by continuity.

We are now in a position to argue precisely as we did in the MF regime. From (4.43) and (4.45) we find as before

$$S+C_1(\Gamma')^*\Gamma'\geq I \text{ on } z=0 \tag{4.46}$$

for some $C_1>0$ provided c was big enough. Continuing we obtain the SF estimate

$$(|U_P|_2^2 + (\gamma+\rho^2)|U_H|_2^2) + |U|^2 \leq C\left(|F_P|_2^2 + \frac{1}{(\gamma+\rho^2)}|F_H|_2^2\right) + C|\Gamma' U|^2, \tag{4.47}$$

uniformly near $(\underline{q}, \underline{\hat{\zeta}}, 0)$. Here we've written $F = F_H + F_P$ in the obvious way and used

$$|(SF, U)| \leq (C_\delta |F_P|_2^2 + \delta |U_P|_2^2) + \left(\frac{C_\delta}{(\gamma+\rho^2)}|F_H|_2^2 + \delta(\gamma+\rho^2)|U_H|_2^2\right) \tag{4.48}$$

to absorb terms from the right.

4.5. **The SF symmetrizer.** The full details of the symmetrizer construction in the SF regime are fairly technical because of the glancing blocks, so we'll omit them in these lectures (see [MZ]). A key point is that once the sign condition is arranged, the details of the construction are quite similar to the classical construction of Kreiss [K] in which there was only a single perturbation parameter (corresponding to our $\hat{\gamma}$). For now we simply record the form that the kth block $S_k(q, \hat{\zeta}, \rho)$ takes when the corresponding block Q_k in (4.27) is, say, a 3×3 glancing block:

$$S_k(q, \hat{\zeta}, \rho) = E_k + \tilde{E}_k(q, \hat{\zeta}) - i\hat{\gamma}F_k - i\rho H_k. \tag{4.49}$$

Here E_k and $\tilde{E}_k$ are real symmetric matrices with $\tilde{E}_k(\underline{q}, \underline{\hat{\zeta}}) = 0$ and

$$E_k = \begin{pmatrix} 0 & 0 & e_{k1} \\ 0 & e_{k1} & e_{k2} \\ e_{k1} & e_{k2} & e_{k3} \end{pmatrix}. \tag{4.50}$$

The matrices F_k and H_k are real and antisymmetric. It is possible to choose S_k of this form to satisfy

$$\text{Re } S_k Q_k \geq \hat{\gamma} D_k + \rho D_k', \text{ where } D_k \geq cI, D_{k'} > cI \tag{4.51}$$

as well as the boundary estimate (4.43)(b) near the basepoint $(\underline{q}, \underline{\hat{\zeta}})$.

Note that the symmetrizers constructed by Kreiss in [K] had a form given by the sum of the first three terms on the right in (4.49).

4.6. **The HF regime.** In HF the spectrum of $\mathcal{G}(q, z, \zeta)$ stays well away from the imaginary axis, so the argument here is similar to the one for MF, except for the extra difficulty that the set of frequencies is noncompact. However, since the parabolic part of the linearized operator is dominant in HF, we can use the natural parabolic homogeneity to reduce to a compact set of parameters. This argument is given in Appendix F.

4.7. **Summary of estimates.** We recall that our goal, established in Lecture 2, has been to prove L^2 estimates for the fully linearized transmission problem (2.25)

$$\begin{aligned}&L_u(q,z,\zeta,\partial_z)v + L_\psi(q,z,\zeta)\phi = f\\ &[v]=0,\ [\partial_z v]=0,\ p(\zeta)\phi - l\cdot v = 0 \text{ on } z=0,\end{aligned} \tag{4.52}$$

which was reformulated as a first order system for the unknowns $U=(v,v_z)$ and ϕ in (2.30):

$$\begin{aligned}&\partial_z U - G(q,z,\zeta)U = F + \begin{pmatrix}0\\ -(B^0(q))^{-1}L_\psi(q,z,\zeta)\phi)\end{pmatrix}\\ &[U]=0,\ p(\zeta)\phi - l(q)\cdot v = 0 \text{ on } z=0\end{aligned} \tag{4.53}$$

In Lecture 3 we defined a good unknown for the problem (4.53), namely $U^\sharp = U - P\phi$, which allows us to reduce the study of (4.53) in MF and HF to the study of (3.35):

$$\partial_z U^\sharp - G(q,z,\zeta)U^\sharp = F,\ \ [U^\sharp]=0. \tag{4.54}$$

In SF the good unknown is $U^\flat = U - R\phi$, and we showed that this allows us to reduce the study of (4.53) to that of (3.43):

$$\begin{aligned}&\partial_z U^\flat - G(q,z,\zeta)U^\flat = F\\ &\pi(q,\hat{\zeta},\rho)[U^\flat]=0,\ l(q)\cdot u_+^\flat = 0.\end{aligned} \tag{4.55}$$

Tracing back through the conjugations, the estimates (4.47) in SF, (4.20) in MF, and (11.25) in HF imply the following estimates for $U^\sharp = (v^\sharp, v_z^\sharp)$ and $U^\flat = (u^\flat, u_z^\flat)$. We define

$$h(\zeta) = \begin{cases}(\gamma+\rho^2)^{1/2},\ \rho\le 1\\ \langle\zeta\rangle,\ \rho>1\end{cases}, \tag{4.56}$$

where

$$\langle\zeta\rangle := (\tau^2+\gamma^2+\eta^4)^{1/4} \sim |\tau,\gamma|^{1/2} + |\eta|. \tag{4.57}$$

Noting that the forcing term F in the above first order systems satisfies

$$|F|_2 \le C|f|_2, \tag{4.58}$$

for f as in (4.52)

We have

$$\begin{aligned}&(a) h^2|u^\flat|_2 + h|u_z^\flat|_2 + h|U^\flat| \le C|f|_2 \text{ in SF}\\ &(b) |v^\sharp|_2 + |v_z^\sharp|_2 + |U^\sharp| \le C|f|_2 \text{ in MF}\\ &(c) \langle\zeta\rangle^2|v^\sharp|_2 + \langle\zeta\rangle|v_z^\sharp|_2 + \langle\zeta\rangle^{3/2}|v^\sharp| + \langle\zeta\rangle^{1/2}|v_z^\sharp| \le C|f|_2 \text{ in HF.}\end{aligned} \tag{4.59}$$

From (3.36) we deduce

$$\langle\zeta\rangle^2|\phi| \le |v^\sharp| \text{ in MF and HF} \tag{4.60}$$

and from (3.41)(b) and the nonvanishing of $[\hat{R}]$ we get

$$|\rho\phi| \le C|U^\flat|. \tag{4.61}$$

In view of (4.60) and (4.61), the three estimates above immediately imply the following estimates for the solution (v, ϕ) to the fully linearized transmission problem (4.52):

$$\begin{aligned}&(a)h^2|v|_2 + h|v_z|_2 + h|v, v_z| + h\rho|\phi| \leq C|f|_2 \text{ in SF}\\ &(b)|v|_2 + |v_z|_2 + |v, v_z| + |\phi| \leq C|f|_2 \text{ in MF}\\ &(c)\langle\zeta\rangle^2|v|_2 + \langle\zeta\rangle|v_z|_2 + \langle\zeta\rangle^{3/2}|v| + \langle\zeta\rangle^{1/2}|v_z| + \langle\zeta\rangle^{7/2}|\phi| \leq C|f|_2 \text{ in HF.}\end{aligned} \tag{4.62}$$

Finally, let's summarize all three estimates in a single estimate. First define

$$h_1(\zeta) = \begin{cases}\rho, \ \rho \leq 1\\ \langle\zeta\rangle^2, \ \rho > 1\end{cases}. \tag{4.63}$$

Then for all ζ we have

$$\begin{aligned}h^2|v|_2 + h|v_z|_2 + h(1+\langle\zeta\rangle)^{1/2}|v| + h(1+\langle\zeta\rangle)^{-1/2}|v_z| + h(1+\langle\zeta\rangle)^{1/2}h_1|\phi|\\ \leq C|f|_2.\end{aligned} \tag{4.64}$$

4.8. **Glancing blocks and glancing modes.**

Definition 4.1. Blocks satisfying condition (iv) in Lemma 4.1 will be referred to as *glancing blocks*. From the earlier discussion of block structure we see these blocks correspond to coalescing eigenvalues of $\hat{H}_\pm$.

In the remainder of this subsection we'll allow our 2D notation to represent any number of space dimensions. Thus, if $\eta \in \mathbb{R}^{d-1}$, when we write $g'(U^0_\pm)i\eta$ we mean

$$g'(U^0_\pm)i\eta := \sum_{j=1}^{d-1} g'_j(U^0_\pm)i\eta_j. \tag{4.65}$$

To define glancing modes and relate them to glancing blocks we consider the linearized inviscid limiting operators

$$\mathbb{L}_\pm(q,\xi,\tau,\eta) = i\tau I + A_\nu(U^0_\pm(0,q), d\psi^0(q))i\xi + g'(U^0_\pm)i\eta \tag{4.66}$$

and the corresponding scalar symbols

$$p_\pm(q,\xi,\tau,\eta) = \det \mathbb{L}_\pm(q,\xi,\tau,\eta). \tag{4.67}$$

Definition 4.2. Define the *glancing set* $\mathcal{G}_q$ to be the set of $(\tau,\eta) \in \mathbb{R}^d \setminus 0$ such that for at least one choice of sign the equation $p_\pm(q,\xi,\tau,\eta) = 0$ has a real root ξ of multiplicity ≥ 2. We'll refer to individual points $(\tau_0,\eta_0) \in \mathcal{G}$ as *glancing modes*.

Clearly, at any point $(\tau_0,\eta_0) \in \mathcal{G}_q$ at least one real root ξ of $p_\pm(q,\xi,\tau,\eta) = 0$, has a branch singularity. (The degree of singularity with respect to τ (η held fixed) is equal to the integer s in (4.70) below.)

The hyperbolicity assumption (H1) implies there exist real functions

$$\tau_1^\pm(q,\xi,\eta), \dots, \tau_m^\pm(q,\xi,\eta),$$

smooth and homogeneous of degree one in $(\xi,\eta)\neq 0$, such that

$$\begin{aligned}&\tau_1^\pm<\cdots<\tau_m^\pm \text{ and}\\&p_\pm(q,\tau,\xi,\eta)=(\tau-\tau_1^\pm(q,\xi,\eta))\cdots(\tau-\tau_m^\pm(q,\xi,\eta)).\end{aligned}\tag{4.68}$$

If $(\tau_0,\eta_0)\in\mathcal{G}_q$, there exist ξ_0 and for at least one choice of sign a $\tau_j^\pm$ (with j uniquely determined by the choice of $\pm$ and (ξ_0,τ_0,η_0)) such that (dropping $\pm$)

$$\begin{aligned}&\tau_0=\tau_j(q,\xi_0,\eta_0),\ \text{and}\\&\partial_\xi\tau_j(q,\xi_0,\eta_0)=0.\end{aligned}\tag{4.69}$$

Moreover, the multiplicity of ξ_0 as a root of $p(q,\xi_0,\tau_0,\eta_0)=0$, and thus the degree of singularity (with respect to τ) of the associated branch point, is equal to s ($2\leq s\leq m$) if and only if

$$\begin{aligned}&\partial_\xi^k\tau_j(q,\xi_0,\eta_0)=0,\ \text{for } k=1,\dots,s-1,\ \text{but}\\&\partial_\xi^s\tau_j(q,\xi_0,\eta_0)\neq 0.\end{aligned}\tag{4.70}$$

Note that this implies at the same time that $\partial_\xi\tau_j(\cdot,\eta_0)$ has no roots nearby ξ_0 other than ξ_0 itself.

To relate glancing modes to glancing blocks note, for example, that

$$p_+(q,\hat\xi,\hat\tau,\hat\eta)=\det\left(A_\nu(U_+^0(0,q),d\psi^0(q))(i\hat\xi-\hat H_+(q,\hat\tau,\hat\gamma=0,\hat\eta,\rho=0)\right).\tag{4.71}$$

So when we have a block of size ν associated to a multiple pure imaginary eigenvalue $i\alpha$ of $\hat H_+(\underline{q},\underline{\hat\zeta},0)$, this means $(\underline{\hat\tau},\underline{\hat\eta})\in\mathcal{G}_{\underline{q}}$ and that α is a root of multiplicity ν of

$$p_+(\underline{q},\xi,\underline{\hat\tau},\underline{\hat\eta})=0.$$

The word *glancing* is used because characteristics $(x(t),t,y(t))$ associated to a glancing mode $(\tau_0,\eta_0)\in\mathcal{G}_q$ with $\tau_0=\tau_j(q,\xi_0,\eta_0)$ satisfy

$$x'(t)=-\partial_\xi\tau_j(q,\xi_0,\eta_0)=0,\tag{4.72}$$

and thus run parallel to the boundary $x=0$.

4.9. **Auxiliary hypothesis for Lecture 5.** This is a good place to state an auxiliary hypothesis that we'll need later. In Lecture 5 we deal with planar inviscid shocks so there is no q dependence anymore. In particular, we can remove the q dependence from all the functions appearing in the previous subsection and in place of (4.66) we have

$$\mathbb{L}_\pm(\xi,\tau,\eta)=i\tau I+A(W_\pm)i\xi+g'(W_\pm)i\eta\tag{4.73}$$

for constant states $W_\pm$.

Clearly, (4.70) and the implicit function theorem imply that for any such (τ_0,ξ_0,η_0) and function τ_j, there exists a function $\xi(\eta)$ such that locally near (ξ_0,η_0)

$$\partial_\xi^{s-1}\tau_j(\xi,\eta)=0 \text{ precisely when } \xi=\xi(\eta).\tag{4.74}$$

Note that $\xi(\eta)$ is smooth and homogeneous of degree one away from $\eta=0$.

We can now state the auxiliary assumption (H4):

Assumption 4.1 (H4)**.** *For any* $(\tau_0, \eta_0) \in \mathcal{G}$*, corresponding root* ξ_0 *of multiplicity* s*, and functions* τ_j *and* $\xi(\eta)$ *as above, we have*

$$\partial_\xi^k \tau_j(\xi(\eta), \eta) = 0 \text{ for } k = 1, \ldots, s-1 \text{ and } \eta \text{ near } \eta_0. \tag{4.75}$$

In other words ξ_0 persists as a root $\xi(\eta)$ of multiplicity s of

$$p(\xi(\eta), \tau_j(\xi(\eta), \eta), \eta) = 0$$

for η near η_0, and (by the remark below (4.70)) there are no other nearby roots of multiplicity > 1.

A compactness argument using the fact that $\mathcal{G}$ is a closed conic set shows that under the assumption (H4) all such branch singularities are confined to a finite union of surfaces

$$\tau = \tau_{j,l}(\eta) \equiv \tau_j(\xi_l(\eta), \eta)$$

on which the singularity (with respect to τ) has order equal to s_l, the multiplicity of the root $\xi_l(\eta)$. We'll usually relabel and replace the double index j, l by a single index as in $\tau = \tau_k(\eta)$. Note that graphs τ_k may well intersect.

Remark 4.1. 1. The statements of this subsection (and the previous one) require only slight modification when the assumption of strict hyperbolicity (H1) is relaxed to the following more general hypothesis of [Z], [GMWZ3]:

(H1'): $f'(u)\xi + g'(u)\eta$ has semisimple real eigenvalues of constant multiplicity for $(\xi, \eta) \in \mathbb{R}^d \setminus 0$ (nonstrict hyperbolicity with constant multiplicity).

In this case the multiplicity of ξ_0 as a root of $p(\xi_0, \tau_0, \eta_0) = 0$ is some integer multiple of s as in (4.70).

2. Condition (H4) is automatic in the cases $d = 1, 2$ and also in any dimension for rotationally invariant problems. In 1D the glancing set is empty. In the 2D case the homogeneity of τ_j and its derivatives implies that the ray through (ξ_0, η_0) is the graph of $\xi(\eta)$ and that (H4) holds there. (H4) also clearly holds if no real root ξ of $p(\xi, \tau, \eta) = 0$ has multiplicity > 2, in particular in the case that all eigenvalues $\tau_j(\xi, \eta)$ are linear or convex/concave in their dependence on ξ.

3. In the equations of gas dynamics all characteristics are linear combinations of (ξ, η) and $|\xi, \eta|$, hence the above results show that (H4) is valid whenever the constant multiplicity assumption (H1') applies. Thus, we see that (H4), though mathematically restrictive, nonetheless allows important physical applications.

4. Precise information on how eigenvalues split near glancing modes (see (5.44)) is important in the constructions of the next lecture. To understand splitting the reader should keep in mind the simple example given in the section on the sign condition.

5. Lecture five: Long time stability via degenerate symmetrizers

In this lecture we focus mainly on obtaining linearized estimates via symmetrizers. Details of the nonlinear endgame are given in [GMWZ1]. There is much overlap with the earlier results of [Z] obtained by construction of Green's functions. However, as described in [GMWZ1] each approach seems to yield some results inaccessible to the other. The symmetrizer approach has the advantage, as we saw in the first four lectures, of applying to curved shocks as well as planar shocks. We note that the nonlinear endgame for Theorem 5.1 is inspired by that in [KK], while the one for

Theorem 5.2 is essentially that of [Z]. A brief discussion of these arguments is given at the end of this Lecture.

Consider the $m \times m$ system of viscous conservation laws

$$u_t + f(u)_x + g(u)_y - \triangle u = 0, \tag{5.1}$$

where now $y \in \mathbb{R}^{d-1}$, $d \geq 2$, but we continue to use 2D notation as in (4.65).

We are given a stationary inviscid shock $x = 0$ with constant states $W_\pm$ and a stationary solution $W(x)$ (the profile) of (5.1) satisfying

$$\begin{aligned} &W_x = f(W) - f(W_-) \\ &\lim_{x \to \pm\infty} W(x) = W_\pm. \end{aligned} \tag{5.2}$$

5.1. **Nonlinear stability.** We wish to understand the stability of the profile $W(x)$ under multidimensional perturbations. Let $\mathcal{A}$ denote some set of admissible perturbations to be specified later.

Definition 5.1. (1) For $v_0 \in \mathcal{A}$ let $u(x,t,y)$ be the solution to the system (5.1) with initial data at $t = 0$ given by

$$u_0(x,y) = W(x) + \delta v_0(x,y). \tag{5.3}$$

We say that W is *nonlinearly stable* with respect to perturbations in $\mathcal{A}$ if there exists a $\delta_0 > 0$ (depending on $|v_0|_{\mathcal{A}}$) such that for $\delta \leq \delta_0$, the solution $u(x,t,y)$ exists for all time and

$$|u(x,t,y) - W(x)|_{L^\infty(x,y)} \to 0 \text{ as } t \to \infty. \tag{5.4}$$

(2) We refer to v_0 as a *zero mass* perturbation if it has the form $v_0 = \text{div } V_0$ for some V_0. General perturbations not necessarily of this form are called *nonzero mass* perturbations.

We look for u of the form

$$u(x,t,y) = W(x) + \delta z(x,t,y). \tag{5.5}$$

To obtain a problem with zero initial data we take

$$z(x,t,y) = v(x,t,y) + e^{-t} v_0(x,y) \tag{5.6}$$

and after a short computation we obtain the following error problem for v, where we set $A = f'$:

(5.7)

$$\begin{aligned} &v_t + (A(W(x))v)_x + g'(W(x))v_y + \delta \text{div}_{x,y}(B(x,t,y)v) + \delta \text{div}_{x,y}(h(x,t,y,v)) - \triangle v = f \\ &v|_{t=0} = 0, \end{aligned}$$

where $h(x,t,y,v) = O(|v|^2)$.

The main step is to prove appropriate estimates for the corresponding linearized problem

$$\begin{aligned} &u_t + (A(W(x))u)_x + g'(W(x))u_y - \triangle u = f \\ &u|_{t=0} = 0. \end{aligned} \tag{5.8}$$

We'll derive these by studying instead the eigenvalue equation, obtained by Fourier-Laplace transform of (5.8) after extending f and u by zero in $t<0$:

$$\begin{aligned} &\hat u_{xx} - (A(W(x))\hat u)_x - s(x,\zeta)\hat u = f \text{ on } \mathbb{R}_x, \text{ where} \\ &s(x,\zeta) = g'(W(x))i\eta + (i\tau+\gamma+\eta^2)I. \end{aligned} \tag{5.9}$$

Dropping the hat on u we set $U=(u,u_x)$ and

$$\begin{aligned} G(x,\zeta) &= \begin{pmatrix} 0 & I \\ \mathcal{M} & A(W)) \end{pmatrix}, \text{ where} \\ \mathcal{M}(x,\zeta) &= s(x,\zeta) + A'(W)\cdot W_x. \end{aligned} \tag{5.10}$$

Next replace (5.9) by the equivalent $4m\times 4m$ doubled boundary problem on $x\geq 0$

$$U_x - \mathcal{G}(x,\zeta)U = F \text{ on } x\geq 0,\ \Gamma U = 0, \tag{5.11}$$

where now $U=(U_+,\tilde U_-)$ (recall (3.15)) and

$$\mathcal{G} = \begin{pmatrix} G & 0 \\ 0 & -\tilde G \end{pmatrix}, \quad \Gamma U = U_+(0) - \tilde U_-(0). \tag{5.12}$$

The final preparatory step is to conjugate to block structure via a conjugator ZYT_cT_B just as in Lecture 4:

$$U_x - \mathcal{G}_B(\hat\zeta,\rho)U = F, \quad \Gamma ZYT_cT_BU := \Gamma^\# U = 0. \tag{5.13}$$

The only difference is that here we retain the degenerate boundary condition $\Gamma U = 0$ in the SF region.

5.2. L^1-L^2 **estimates.** The main step is to establish L^1-L^2 estimates in SF for (5.13), assuming now the auxiliary Assumption 4.1 (or (H4)). That is, we assume that branch singularities of characteristic roots ξ (considered as functions of (τ,η)) are confined to a finite union of smooth surfaces $\tau=\tau_j(\eta)$ on which the singularity has constant order equal to s_j, the multiplicity of the corresponding root ξ.

In MF and HF we can use the estimates established in Lecture 4. The following Proposition is proved in the next few subsections.

Proposition 5.1. *Fix a basepoint $X_0=(\underline{\hat\zeta},0)$. Assume (H1),(H2),(H3),(H4), and*

$$-\theta\rho(\hat\tau^2+\hat\eta^2) \leq \hat\gamma \leq C\rho \tag{5.14}$$

for some $C>0$ and small enough $\theta>0$.

Then, for $F\in L^1$ and $\rho>0$ sufficiently small, the solution of the conjugated doubled boundary problem (5.13) *satisfies*

$$|U|^2_{L^2(x)} \leq \frac{C\beta^2|F|^2_{L^1(x)}}{\rho^2} \tag{5.15}$$

for some $C>0$ uniformly near the basepoint X_0, where

$$\beta(\hat\zeta,\rho) := \max_{j\geq 0}\beta_j, \tag{5.16}$$

with $\beta_0:=1$ and

$$\beta_j := (|\hat\tau-\tau_j(\hat\eta)|+\rho+\hat\gamma)^{1/s_j-1}. \tag{5.17}$$

(Note that $\beta = 1$ if the glancing set $\mathcal{G}$ is empty, in particular for $d = 1$.)

From (5.15) together with the MF and HF estimates from Lecture 4, we obtain readily the following linear estimate. This estimate leads directly to a proof of long time stability for nonzero mass perturbations in space dimensions $d \geq 3$.

Corollary 5.1. *Assume (H1), (H2), (H3), and (H4). Then, for $d \geq 3$, the solution of the linear problem* (5.8) *(nonzero mass) satisfies*

$$|u, u_y, u_t|_{L^2(x,t,y)} + |u_x|_{L^2(x,t,y)} \leq C(|f|_{L^1(x,t,y)} + |f|_{L^2(x,t,y)}). \tag{5.18}$$

Proof. We'll write $|F|_{L^1(x)} = |F|_1$.

Define V and H by $U = \hat{V}(x, \tau, \gamma, \eta)$, $F = \hat{H}$, where $0 < \gamma \leq C\rho^2$, and suppose now that U and F are supported in $\rho < \delta$.

(5.15) gives

$$|U|_2^2 \leq \frac{C\beta^2}{|\tau, \eta|^2}|F|_1^2. \tag{5.19}$$

Integrate (5.19) $d\tau d\eta$ (dimension of (τ, η) space is ≥ 3) to get

$$|e^{-\gamma t}V|^2_{L^2(x,t,y)} \leq \int \frac{C\beta^2}{|\tau, \eta|^2}|\hat{H}(x, \tau, \gamma, \eta)|^2_{L^1(x)} d\tau d\eta. \tag{5.20}$$

But

$$|\hat{H}(x, \tau, \gamma, \eta)| \leq C|H(x, t, y)|_{L^1(t,y)}, \tag{5.21}$$

so

$$|\hat{H}(x, \tau, \gamma, \eta)|_{L^1(x)} \leq C|H(x, t, y)|_{L^1(x,t,y)}. \tag{5.22}$$

Plug this into (5.20) to get

$$|e^{-\gamma t}V|^2_{L^2(x,t,y)} \leq \int_{|\tau,\eta|<\delta} \frac{C\beta^2}{|\tau, \eta|^2}|H|^2_{L^1(x,t,y)} d\tau d\eta \leq C|H|^2_{L^1(x,t,y)}. \tag{5.23}$$

Here we used the fact that for $d \geq 3$

$$\int_{|\tau,\eta|<\delta} \frac{\beta^2}{|\tau, \eta|^2} d\tau d\eta < \infty. \tag{5.24}$$

We note that a little care is needed in showing this since β is singular.

Finally, let $\gamma \to 0$ to get

$$|V|^2_{L^2(x,t,y)} \leq C|H|^2_{L^1(x,t,y)}. \tag{5.25}$$

For U and F supported in MF or HF, the results of Lecture 4 yield estimates with only the L^2 norm of H on the right. The corollary follows.

□

Note that (5.24) fails for $d = 1, 2$ and some different ideas are needed. For $d = 1$ we refer to [ZH]. For $d = 2$ we need $L^1 - L^p$ estimates for (5.13).

In what follows we'll occasionally interpolate between L^2 and L^∞ using the following elementary inequalities:

$$|u|_{L^p} \le |u|_{L^\infty}^{1-\frac{2}{p}} |u|_{L^2}^{\frac{2}{p}} \le |u|_{L^\infty} + |u|_{L^2}. \tag{5.26}$$

From (5.15) we obtain readily the following $L^1 \to L^p$ bounds.

Corollary 5.2. *Assume (H1), (H2), (H3), (H4), and* (5.14). *Then, for $F \in L^1$ and $\rho > 0$ sufficiently small, the solution of the conjugated doubled boundary problem* (5.13) *satisfies*

$$|u|_{L^p} \le \frac{C\beta |F|_{L^1}}{\rho} \tag{5.27}$$

for all $2 \le p \le \infty$, for some $C > 0$ uniformly near the basepoint X_0, where β is defined as in Proposition 5.1.

Proof. Recall that $|U|$ bounds both $|u|$ and $|u_x|$. Thus, the result for $p = \infty$ follows from the standard one-dimensional Sobolev inequality

$$|f|_\infty \le |f|_2^{1/2} |f_x|_2^{1/2}, \tag{5.28}$$

and the general result $2 \le p \le \infty$ by interpolation between L^2 and L^∞ norms. □

5.3. **Proof of Proposition 5.1.** Our strategy in proving Proposition 5.1 will be to establish an $L^2 \to L^\infty$ bound for the adjoint problem, then appeal to duality. In deriving adjoint $L^2 \to L^\infty$ bounds, we use duality in a second way, to first conclude adjoint $L^2 \to L^2$ bounds from the $L^2 \to L^2$ bounds of the forward equation (slightly refined). From the adjoint L^2 bounds, $L^2 \to L^\infty$ bounds are then readily obtained by a standard energy estimate/integration by parts.

Remark 5.1. It is worth noting that we do not in this argument apply degenerate symmetrizers to the adjoint equation. Indeed, because of an asymmetry between forward vs. dual equations, our standard degenerate symmetrizer estimate would not recover the sharp bound available by duality. (Specifically, the degeneracy in the boundary condition for the dual equation occurs in hyperbolic modes, though we shall not show it here.)

5.4. **The dual problem.** Consider a general boundary problem

$$\begin{aligned} LU &:= U_x - G(x,\zeta)U = F \\ \Gamma U &= 0 \text{ on } x = 0. \end{aligned} \tag{5.29}$$

The dual problem is then defined via L^2 inner product on $\mathbb{R}^+$ as

$$\begin{aligned} L^*V &:= -V_x - G^*(x,\zeta)U = G \\ \Gamma^* V &= 0 \text{ on } x = 0, \end{aligned} \tag{5.30}$$

where the kernel of Γ^* is the orthogonal complement of the kernel of Γ, i.e., by the property that

$$\langle LU, V\rangle = \langle U, L^*V\rangle \tag{5.31}$$

for $\Gamma U(0) = \Gamma^* V(0) = 0$.

A formality is to first establish well-posedness of both problems.

Proposition 5.2. *For $\rho > 0$, both forward and dual problems have a unique H^1 solution for any data in L^2.*

Proof. It is sufficient to prove uniqueness, which follows in both cases from the *standard* (nondegenerate) symmetrizer construction carried out for fixed $\rho \neq 0$. The interior estimates thereby obtained feature constants that may blow up arbitrarily fast in ρ as $\rho \to 0$; however, this is of no consequence for the present purpose. □

Corollary 5.3. *The bound of Proposition 5.1 is equivalent to the dual bound*

$$|V|^2_{L^\infty} \leq \frac{C\beta^2}{\rho^2}|G|^2_2 \tag{5.32}$$

for solutions of the dual conjugated boundary problem, for $G \in L^2$.

Proof. We have

$$|U|_{L^2} = \sup_{|G|_{L^2}=1} \langle U, G\rangle = \langle U, L^*V\rangle = \langle LU, V\rangle = \langle F, V\rangle \leq |F|_{L^1}|V|_{L^\infty}, \tag{5.33}$$

from which we obtain the forward direction

$$|U|_{L^2}/|F|_{L^1} \leq |V|_{L^\infty}/|G|_{L^2}. \tag{5.34}$$

A reverse calculation yields the backward direction. □

5.5. **Decomposition of $U_{H\pm}$.** To establish (5.32), we will need to sharpen the basic $L^2 \to L^2$ estimate for the forward equation. To do this, we shall need to decompose the hyperbolic modes U_H in decomposition (4.38) as the sum $U_H = U_{H_+} + U_{H_-}$, where

$$U_{H_\pm} = U_{H_{h\pm}} + U_{H_{e\pm}} + U_{H_{g\pm}}. \tag{5.35}$$

Each vector appearing in (5.35) has $4m$ components, and the decomposition depends on $(\hat{\zeta}, \rho)$. While U_H here is the same as the vector U_H appearing in (4.38), to avoid confusion it is important to note that the definitions of $U_{H_\pm}$ are different now as we explain below.

We shall write

$$U_{H_h} = U_{H_{h+}} + U_{H_{h-}}$$

and do similarly for e and g. The hyperbolic mode $U_{H_{h\pm}}$ has nonvanishing components corresponding (only) to the blocks Q_k in (4.27) which are 1×1 with real part vanishing at the base point, but with real part > 0 (resp.< 0) when $\rho > 0$. The elliptic mode $U_{H_{e\pm}}$ has nonvanishing components corresponding to blocks with $\Re Q_k$ positive or negative definite at the base point. Finally, the glancing mode U_{H_g} has nonvanishing components corresponding to blocks of size larger than 1×1 which are purely imaginary at the base point (glancing blocks).

Further, we shall diagonalize the glancing blocks by a $4m \times 4m$ matrix $T_{H_g}(\hat{\zeta}, \rho)$:

$$U'_{H_g} := T^{-1}_{H_g} U_{H_g}, \tag{5.36}$$

where $U_{H_g} := U_{H_{g+}} + U_{H_{g-}}$. Here $U_{H_{g\pm}}$ are defined as the projections of U_{H_g} onto the growing (resp. decaying) eigenspaces of $\hat{H}_B$ in (4.27) corresponding to glancing blocks. Call these subspaces $H_{g\pm}$. Clearly, T_{H_g} also has a block structure and we may construct it so that in any given block corresponding to a glancing block $\mathcal{Q}_j$, the first columns are eigenvectors of $\mathcal{Q}_j$ associated (for $\rho > 0$) to eigenvalues with $\Re\mu < 0$. The remaining blocks of T_{H_g} are identity matrices.

We denote by

$$U' := T_{H_g}^{-1} U \tag{5.37}$$

the full variable with U_{H_g} diagonalized, and all other components unchanged. By calculations similar to those in [Z], we obtain the following estimates.

Lemma 5.1. *The diagonalizing transformation T_{H_g} may be chosen so that*

$$|T_{H_g}| \le C, \tag{5.38}$$

$$|T_{H_g}^{-1}| \le C\beta, \tag{5.39}$$

and

$$|T_{H_g}^{-1}{}_{|H_{g-}}| \le C\alpha, \tag{5.40}$$

where $\beta := \max_j \beta_j$, $\alpha := \max_j \alpha_j$, with

$$\beta_j := \theta_j^{1-s_j}, \qquad \alpha_j := \theta_j^{1-[(s_j+1)/2]}, \tag{5.41}$$

$$\theta_j := (|\hat\tau - \tau_j(\hat\eta)| + \hat\gamma + \rho)^{1/s_j}, \tag{5.42}$$

and $T_{H_g}^{-1}{}_{|H_{g-}}$ denotes the restriction of $T_{H_g}^{-1}$ to subspace H_{g-}. In particular,

$$\beta\alpha^{-2} \ge 1. \tag{5.43}$$

Remark 5.2. The quantities β and α, and their sharp estimation, we regard as a key to the analysis of long-time stability in multidimensions.

Proof. Clearly, it is sufficient to establish for a single block Q_j of size s_j that there exist diagonalizing matrices whose inverses are bounded by β_j, α_j, respectively. Let $\underline{\mu}$ denote the multiple pure imaginary eigenvalue appearing in Q_j evaluated at the basepoint $(\underline{\hat\tau}, \underline{\hat\eta})$. From here on, we drop the j subscript.

Set $\sigma = |\hat\tau - \tau(\hat\eta)| + \hat\gamma$ so $\theta = (\sigma+\rho)^{1/s}$. By a classic matrix perturbation argument (e.g., [Z], Lemma 4.8) the eigenvalue $\underline{\mu}$ splits for $\sigma + \rho > 0$ small into s eigenvalues.

$$\mu_k = \underline{\mu} + \pi_k + o(|\sigma, \rho|^{1/s}),\ k = 1, \dots, s \tag{5.44}$$

Here

$$\begin{aligned} &\pi_k = \epsilon^k i(p\sigma - iq\rho)^{1/s} \text{ with} \\ &\epsilon = 1^{1/s}, \\ &p(\hat\eta) \text{ and } q(\hat\eta) \text{ are real and } \sim 1, \text{and sgn } p = \text{sgn } q. \end{aligned} \tag{5.45}$$

Moreover, correponding eigenvectors are given in appropriate coordinates by

$$(1, \pi_k, \pi_k^2, \dots, \pi_k^{s-1}) + o(|\sigma, \rho|^{1/s}). \tag{5.46}$$

Thus, there exists a matrix T_{H_g} of eigenvectors of the $s \times s$ block Q that is approximately given by a vandermonde matrix with generators distance at least θ apart related by s roots of unity.

By Kramer's rule, we may therefore estimate β as the quotient of two vandermonde determinants, the numerator of size $s-1$ and the denominator of size s, taken from the same set of equally spaced generators. The standard formula for vandermonde determinants gives then

$$\beta \sim \theta^{\binom{s-1}{2} - \binom{s}{2}} = \theta^{1-s} \tag{5.47}$$

as claimed.

Denoting by

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \tag{5.48}$$

the matrix consisting of the $k \leq [(s+1)/2]$ stable eigenvectors of Q, i.e., the first k columns of T_{H_g}, and noting that t_1 as a vandermonde matrix is invertible, we find that H_{g-} consists of vectors of form

$$\begin{pmatrix} w \\ t_2 t_1^{-1} w \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} t_1^{-1} w, \tag{5.49}$$

where $w \in \mathbb{C}^k$ is arbitrary.

From $|(w, t_2 t_1^{-1} w)| \geq |w|$ and the computation

$$\begin{aligned} \left| T_{H_g}^{-1} \begin{pmatrix} w \\ t_2 t_1^{-1} w \end{pmatrix} \right| &= \left| \begin{pmatrix} t_1 & * \\ t_2 & * \end{pmatrix}^{-1} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} t_1^{-1} w \right| \\ &= \left| \begin{pmatrix} I_k \\ 0 \end{pmatrix} t_1^{-1} w \right| \\ &= |t_1^{-1} w| \end{aligned} \tag{5.50}$$

we thus obtain that $|T_{H_g}^{-1}{}_{|H_{g-}}| \leq |t_1^{-1}|$.

Observing that t_1 is a $k \times k$ vandermonde matrix with generators taken from the same equally spaced set, and applying Kramer's rule similarly as before, we obtain

$$|t_1^{-1}| \leq C\theta^{1-[(s+1)/2]}, \tag{5.51}$$

and thus $\alpha = \theta^{1-[(s+1)/2]}$ as claimed. □

We define similar decompositions on the dual variable V, and also the forcing terms F and G.

5.6. **Interior estimates.** We begin by carrying out a basic degenerate symmetrizer estimate for the diagonalized forward problem. Note that the treatment of glancing modes is considerably simpler in diagonalized coordinates, and indeed has nothing to do with that of the original Kreiss construction.

Lemma 5.2. *For the forward diagonalized problem, we have the interior bound*

$$|U'|_{L^2}^2 \leq C \frac{|F'|_{L^2}^2}{\rho^2(\gamma + \rho^2)}. \tag{5.52}$$

Proof. In diagonalized coordinates, we must deal with a new degeneracy of order α^{-1} in the glancing modes of the diagonalized boundary condition $\Gamma' := \Gamma T_{H_g}$ for the forward problem, as may be seen by the calculation

$$|\Gamma' U'_{H_{g-}}| = |\Gamma U_{H_{g-}}| \geq C^{-1}|U_{H_{g-}}| \geq \frac{C^{-1}|U'_{H_{g-}}|}{|T^{-1}_{H_g\,|H_{g-}}|}. \tag{5.53}$$

On the other hand, there are no coalescing modes, and so we may dispense with the usual Kreiss construction, treating glancing modes in the same way as hyperbolic and elliptic modes. Precisely, in each S_H block except for those corresponding to glancing modes, we make the same choice of nondegenerate symmetrizer as in Lecture 4, while for each glancing blocks we choose a degenerate symmetrizer

$$S_{H_g} = diag(S_{H_{g+}}, S_{H_{g-}}) := diag(C, -\alpha^{-2}) \tag{5.54}$$

(recall, $\alpha^{-1} \to 0$ as $\sigma + \rho \to 0$).

In view of the glancing degeneracy (5.53) and the translational degeneracy (which we have not removed in the long time problem), there holds

$$|\Gamma' U'_-| \geq C(\delta|U'_{H_{h-}}| + \delta|U'_{H_{e-}}| + \alpha^{-1}|U'_{H_{g-}}| + \rho|U'_{P-}|). \tag{5.55}$$

So if we take the S_P also to be degenerate

$$S_P = \begin{pmatrix} cI & 0 \\ 0 & -\rho^2 \end{pmatrix}, \tag{5.56}$$

we again obtain good trace terms in the resulting symmetrizer estimate.

It remains to check that we retain good interior (L^2) bounds. Let $\mu_{k\pm}$ denote the eigenvalue associated with the kth mode of U'_{H_g}. Taylor expanding the expression (5.45) for π_k about $\rho/\sigma = 0$ yields,

$$|\Re\mu_{k\pm}| \geq C^{-1}\rho^2\beta, \tag{5.57}$$

whence we obtain from the fact that $\beta\alpha^{-2} \geq 1$ the lower bound

$$|\Re\mu_{k\pm}| \geq C^{-1}\alpha^2\rho^2, \tag{5.58}$$

and thereby the key interior estimate

$$(\text{Re } SG'_B(\infty)U'_{H_g}, U'_{H_g}) \geq \alpha^2\rho^2|U'_{H_{g+}}|_2^2 + \rho^2|U'_{H_{g-}}|_2^2. \tag{5.59}$$

That is, we still find that $\Re SG'_B(\infty) \geq \rho^2$, and therefore the rest of the symmetrizer argument goes through as before to give the claimed estimate. □

Remark 5.3. Since T_{H_g} diagonalizes the forward problem, $T^{-1*}_{H_g}$ diagonalizes the dual problem.

By duality, this yields

Corollary 5.4. *For the dual diagonalized problem, we have the interior bound*

$$|V'|^2_{L^2} \leq \frac{C|G'|^2_{L^2}}{(\gamma + \rho^2)\rho^2}. \tag{5.60}$$

In fact, the above estimates can be somewhat refined. Let $U'_{H_{g\pm,j}}$ denotes the jth growing/decaying glancing mode, and $\mu_{j\pm}$ the associated growth/decay rate (eigenvalue of G_B).

Lemma 5.3. *For the forward diagonalized problem, we have the refined interior bounds*

$$|U'|_2^2 \le C\frac{|F_P|_2^2 + (\gamma+\rho^2)^{-1}|F_{H_h}|_2^2 + \rho^{-1}|F_{H_e}|_2^2 + \sum_{j,\pm}|\Re\mu_{j\pm}|^{-1}|F_{H_{g\pm,j}}|_2^2}{\rho^2}. \tag{5.61}$$

Proof. Parabolic modes have growth/decay rates with real part bounded in absolute value above and below by order one; elliptic modes have growth/decay rates bounded above and below by order ρ; hyperbolic modes have growth/decay rates bounded above and below by order $(\gamma+\rho^2)$. Glancing modes are treated individually in the diagonalized coordinates, and have growth/decay rates with absolute value of real part $|\Re\mu_{j\pm}|$. Using this sharp information in the degenerate symmetrizer estimate described just above, specifically in the application of Young's inequality to estimate $|(SF', U')|$ we obtain the claimed bound. □

Corollary 5.5. *For the dual diagonalized problem, we have the interior bounds*

$$|V'_P|_{L^2}^2 + (\gamma+\rho^2)|V'_{H_h}|_{L^2}^2 + \rho|V'_{H_e}|_{L^2}^2 + \sum_{j\pm}|\Re\nu_{j\pm}||V'_{H_{g\pm,j}}|_{L^2}^2) \le \frac{C|G'|_{L^2}^2}{\rho^2}, \tag{5.62}$$

where $\nu_{j\pm} = -\mu^*_{j\mp}$ denote growth/decay rates for the dual problem (eigenvalues of $-G_B^*$).

Proof. Integration by parts, exactly as in the proof of Corollary 5.3, but mode by mode. For example, to obtain the bound

$$\rho|V'_{H_e}|_{L^2}^2 \le \frac{C|G'|_{L^2}^2}{\rho^2}, \tag{5.63}$$

we begin with bound

$$\rho|U'|_{L^2}^2 \le C\rho^{-2}|F'_{H_e}|_{L^2}^2 \tag{5.64}$$

for the forward problem $L'U' = F'_{H_e}$, and calculate

$$|V'_{H_e}|_{L^2} = \sup_{|F'_{H_e}|=1}\langle V'_{H_e}, F'_{H_e}\rangle = \sup\langle V', L'U'\rangle = \sup\langle L'^*V', U'\rangle \tag{5.65}$$

$$= \sup|G'|_{L^2}|U'|_{L^2} \le |G'|_{L^2}C\rho^{-3/2}|F'_{H_e}|_{L^2} = C\rho^{-3/2}|G'|_{L^2}. \tag{5.66}$$

□

5.7. L^∞ **estimates.** With these preparations, L^∞ estimates are now easily obtained.

Lemma 5.4. *For the dual problem, we have the bounds*

$$|V'|_\infty^2 \le \frac{C|G'|_{L^2}^2}{\rho^2}, \qquad |V|_\infty^2 \le \frac{C\beta^2|G|_{L^2}^2}{\rho^2}. \tag{5.67}$$

Proof. Working in diagonalized coordinates, we may take the real part of the L^2 inner product of V' with equation $(L')^*V' = G'$ from $x_0 \geq 0$ to plus infinity to obtain after integration by parts the estimate

$$\begin{aligned}|V'(x_0)|^2 \leq C(|V'_P|_2^2 + (\gamma+\rho^2)|V'_{H_h}|_2^2 + \rho|V'_{H_e}|_2^2 + \sum_{j\pm} |\Re\nu_j||V'_{H_{g\pm,j}}|_2^2) \\ + C|V'|_2|G'|_2.\end{aligned} \tag{5.68}$$

Bounding the first term on the righthand side using Corollary 5.5 and the second term using Corollary 5.4, we obtain the first asserted bound. The second asserted bound then follows by change of coordinates and the Jacobian bounds of Lemma 5.1. □

This completes the proof of Proposition 5.1.

5.8. **Nonlinear stability results.** Recall from Definition 5.1 the definition of nonlinear stability with respect to a given family of perturbations $\mathcal{A}$. Define

$$\begin{aligned}\mathcal{A}_p &= \{v_0(x,y) : v_0 \in W^{p+2,2} \cap W^{2,1}\} \\ \mathcal{A}_\infty &= \{v_0(x,y) : v_0 \in L^\infty \cap L^1\},\end{aligned} \tag{5.69}$$

where $W^{k,s}$ is the standard Sobolev space (k is order of differentiation; s is the L^s exponent).

Theorem 5.1 (nonzero mass, $d \geq 3$). *Assume (H1),(H2),(H3),(H4) and $p > \frac{d}{2}$, where the number of space dimensions is $d \geq 3$. Then the viscous profile $W(x)$ is nonlinearly stable with respect to $\mathcal{A}_p$.*

This theorem follows from the linear estimate of Corollary 5.1 and the nonlinear endgame of [KK]. The idea is to use the linear estimate together with standard Sobolev and Moser inequalites to show that the perturbation v as in (5.7) satisfies

$$|v|_{W^{p+1,2}(T)} + |v_t|_{W^{p,2}(T)} \leq E \tag{5.70}$$

for a fixed E independent of $[0,T]$. This implies that $|v|_{L^\infty(x,y)}$ decays to zero as $t \to \infty$.

The proof of Corollary 5.1 does not work when $d = 2$ since β^2/ρ^2 is not integrable then. This reflects the underlying fact that the linearized response to nonzero mass L^1 initial data in general decays in L^p, $p \geq 2$ no faster than a $d-$dimensional heat kernel.

The endgame in dimension 2 seems to require a special argument similar to the one in [Z]. The corresponding nonlinear stability result is Theorem 5.2. Here, the inverse Laplace transform of the solution to the linearized error problem is estimated via an integral on a parabolic contour $\gamma = -\theta|\tau,\eta|^2$ rather than the flat contour $\gamma = 0$, to take into account the additional decay due to diffusion in the parabolic case. The main ingredient for this argument is the $L^1 - L^p$ estimate of Corollary 5.2.

Theorem 5.2 (nonzero mass, $d \geq 2$). *Assume (H1),(H2),(H3), and (H4), where the number of space dimensions is $d \geq 2$. Then the viscous profile $W(x)$ is nonlinearly stable with respect to $\mathcal{A}_\infty$. Moreover, the perturbation v decays in L^p, $p \geq 2$ at the rate $|v|_p(t) \leq C(p,d)(1+t)^{-\frac{d-1}{2}(1-\frac{1}{p})}$ of a $(d-1)$-dimensional heat kernel, where $C(p,d)$ is monotone increasing in p, finite for $p < \infty$, and uniformly bounded for $d \geq 3$.*

6. Appendix A: The uniform stability determinant

Consider the homogeneous version of the linearized inviscid shock problem

$$\begin{aligned}&\partial_t v + A_\nu(U^0, d\psi^0)\partial_x v + g'(U^0)\partial_y v = 0 \text{ in } \pm x \geq 0\\&\phi_t[U^0] + \phi_y[g(U^0)] - [A_\nu(U^0, d\psi^0)v] = 0 \text{ on } x = 0.\end{aligned} \tag{6.1}$$

We've already encountered this problem (with nonzero forcing) in the construction of higher profiles (1.29). To obtain a stability condition we freeze $q = (t, y)$ in $(U^0(0, t, y), \psi^0(t, y))$, Fourier-Laplace transform in t and Fourier transform in y to get (with $\zeta = (\tau, \gamma, \eta)$, $\gamma \geq 0$)

$$\begin{aligned}&(i\tau + \gamma)\hat{v} + A_\nu\partial_x\hat{v} + g'(U^0)i\eta\hat{v} = 0\\&(i\tau + \gamma)\hat{\phi}[U^0] + i\eta\hat{\phi}[g(U^0)] - [A_\nu\hat{v}] = 0,\end{aligned} \tag{6.2}$$

or, rearranging a little,

$$\begin{aligned}&\partial_x\hat{v} - \mathbb{H}(q, \zeta)\hat{v} = 0 \text{ in } \pm x \geq 0\\&\hat{\phi}\left((i\tau + \gamma)[U^0] + i\eta[g(U^0)]\right) - [A_\nu\hat{v}] = 0 \text{ on } x = 0, \text{ where}\\&\mathbb{H}_\pm(q, \zeta) := -A_\nu(U^0_\pm, d\psi^0)^{-1}\left((i\tau + \gamma)I + i\eta g'(U^0_\pm)\right).\end{aligned} \tag{6.3}$$

For $\gamma > 0$ the negative (resp. positive) generalized eigenspace of $\mathbb{H}_\pm$ has dimension k (resp. l), varies smoothly with (q, ζ), and extends *continuously* to $\gamma \geq 0$ in $\{\zeta \neq 0\}$. Here negative/positive refers to $\Re\mu$. To see the dimensions are correct use (H1), set $(\tau, \eta) = 0$, and note the minus sign in the definition of $\mathbb{H}$. Continuous extensions of decaying eigenspaces are discussed in Appendix B.

Thus, we may choose bases $\{r^1_+(q, \zeta), \dots, r^k_+\}$ and $\{r^1_-(q, \zeta), \dots, r^l_-\}$ for these spaces, where the $r^j_\pm$ are homogeneous of degree zero in ζ and have the same regularity (as the spaces) in (q, ζ) for $\zeta \neq 0$. Clearly, there will be unstable modes growing exponentially with time if for some $\gamma > 0$ the m vectors

$$(i\tau + \gamma)[U^0] + i\eta[g(U^0)], A_\nu(U^0_+, d\psi^0)r^s_+, A_\nu(U^0_-, d\psi^0)r^t_- \quad (s = 1, \dots, k; t = 1, \dots, l) \tag{6.4}$$

are linearly dependent.

As before we let $S^2_+ = \{\hat{\zeta} : |\hat{\zeta}| = 1, \hat{\gamma} \geq 0\}$. The inviscid shock $(U^0(0, q), \psi^0(q))$ is *uniformly stable* if for all q the $m \times m$ determinant

$$\Delta(q, \hat{\zeta}) := \det\left((i\hat{\tau} + \hat{\gamma})[U^0] + i\hat{\eta}[g(U^0)], A_\nu(U^0_+, d\psi^0)r^{(s)}_+(q, \hat{\zeta}), A_\nu(U^0_-, d\psi^0)r^{(t)}_-(q, \hat{\zeta})\right) \tag{6.5}$$

is nonvanishing (here (s) indicates k columns).

In [M2] Majda showed that uniform stability implies optimal L^2 estimates for the linearized problem. In [M3] he used those estimates to construct curved multiD shocks.

7. Appendix B: Continuity of decaying eigenspaces

In Lecture 3 we defined

$$\mathbb{E}(q,\hat{\zeta},\rho) := E_+(q,\hat{\zeta},\rho) \times E_-(q,\hat{\zeta},\rho), \tag{7.1}$$

the decaying generalized eigenspace for $U_z - \mathcal{G}(q,z,\zeta)U = 0$ on $z \geq 0$. We know that $E_\pm$ are C^∞ functions of their arguments in $\hat{\gamma} + \rho > 0$. Here we show

Proposition 7.1. *$E_\pm(q,\hat{\zeta},\rho)$ extend continuously to the corner $\hat{\gamma} + \rho = 0$.*

Proof. **1.** We work near a basepoint $\underline{X} = (\underline{q}, \hat{\underline{\zeta}}, 0)$. In Lecture 4 we saw that the change of variables $U = ZYT_cT_BV$ reduces the study of $U_z - \mathcal{G}(q,z,\zeta)U = 0$ near $\underline{X}$ to that of $V - \mathcal{G}_B(q,\hat{\zeta},\rho)V = 0$, where $\mathcal{G}_B(q,\hat{\zeta},\rho)$ is the block structure matrix:

$$\begin{gathered}\mathcal{G}_B(q,\hat{\zeta},\rho) = \begin{pmatrix} H_B(q,\hat{\zeta},\rho) & 0 & 0 \\ 0 & P_g(q,\zeta) & 0 \\ 0 & 0 & P_d(q,\zeta) \end{pmatrix} \text{ with} \\ \hat{H}_B(q,\hat{\zeta},\rho) = \begin{bmatrix} Q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_s \end{bmatrix}\end{gathered} \tag{7.2}$$

for Q_j as in (4.27). We'll show that $\mathbb{E}$ extends continuously to the corner, but the same block by block argument shows the individual factors $E_\pm$ extend continuously as well.

2. Let $\mathbb{F}(q,\hat{\zeta},\rho)$ be the decaying generalized eigenspace for $\mathcal{G}_B$ in $\hat{\gamma} + \rho > 0$. It suffices to obtain a continuous extension of $\mathbb{F}$ to the corner. In the obvious way we write

$$\mathbb{F}(q,\hat{\zeta},\rho) = (\oplus_{j=1}^s \mathbb{F}_j) \oplus \mathbb{F}_g \oplus \mathbb{F}_d, \tag{7.3}$$

where, for example, $\mathbb{F}_g = \{0\}$ and $\mathbb{F}_d = \mathbb{C}^{m+1}$ in $\hat{\gamma} + \rho > 0$, each having a smooth extension to the corner.

If Q_j is a block of size ν_j satisfying $\pm\Re Q_j > 0$, we have $\mathbb{F}_j = \{0\}$ (resp. $\mathbb{C}^{\nu_j}$) in $\hat{\gamma} + \rho > 0$, so again there is a smooth extension to the corner.

If $\nu_j = 1$ we use the sign condition to see that $\mathbb{F}_j = \{0\}$ or $\mathbb{C}$ in $\hat{\gamma}+\rho > 0$, depending on whether the common sign of $\partial_{\hat{\gamma}}\Re Q_j$ and $\partial_\rho \Re Q_j$ is positive or negative. Here too we have a smooth extension.

3. There remains the case of a glancing block $Q_j(X)$ of size $\nu_j > 1$, where $X = (q,\hat{\zeta},\rho)$ and $Q_j(\underline{X})$ has the Jordan form (4.28). Here we follow an argument in Chapter 7 of [CP].

Setting $Q_j = Q$, $\mathbb{F}_j = \mathbb{F}_-$, $i\alpha_j = i\alpha$, and $\nu_j = \nu$, for $\hat{\gamma} + \rho > 0$ we may write

$$\det(\xi - Q(X)) = \prod_{\Re\mu_k > 0} (\xi - \mu_k)^{\beta_{k+}} \prod_{\Re\mu_k < 0} (\xi - \mu_k)^{\beta_{k-}} := p_+(\xi, X)p_-(\xi, X), \tag{7.4}$$

where again the sign condition implies the numbers

$$\beta_\pm = \sum_k \beta_{k\pm} \tag{7.5}$$

are independent of X for X close to $\underline{X}$ and $\hat{\gamma}+\rho>0$. Note that

$$\mathbb{F}_-(X)=\oplus_{\Re\mu_k<0}\ \ker(Q(X)-\mu_k)^{\beta_{k-}} \tag{7.6}$$

and define

$$\mathbb{F}_+(X)=\oplus_{\Re\mu_k>0}\ \ker(Q(X)-\mu_k)^{\beta_{k+}}, \tag{7.7}$$

so

$$\mathbb{F}_+(X)\oplus\mathbb{F}_-(X)=\mathbb{C}^\nu \text{ for } \hat{\gamma}+\rho>0. \tag{7.8}$$

If we define matrices $\mathcal{P}_\pm(X)=p_\pm(Q(X),X)$, we have in view of the decomposition (7.8)

$$\mathbb{F}_\pm=\ker\mathcal{P}_\pm,\ \text{rank}\ \mathcal{P}_\pm=\dim\mathbb{F}_\mp. \tag{7.9}$$

The Cayley-Hamilton theorem implies $\mathcal{P}_-\mathcal{P}_+=0$, so

$$\text{range}\ \mathcal{P}_+(X)\subset\ker\mathcal{P}_-(X)=\mathbb{F}_-(X). \tag{7.10}$$

Thus, $\mathbb{F}_-(X)=\text{image}\ \mathcal{P}_+(X)$ for $\hat{\gamma}+\rho>0$.

4. To define a continuous extension of $\mathbb{F}_-$ we first use continuity of eigenvalues to extend $\mathcal{P}_+(X)$ continuously to $\mathcal{P}_+(\underline{X})$ by defining

$$\mathcal{P}_+(\underline{X})=(Q(\underline{X})-i\alpha)^{\beta_+}=(Q(\underline{X})-i\alpha)^{\nu-\beta_-}. \tag{7.11}$$

We then define $\mathbb{F}_-(\underline{X})=\text{image}\ \mathcal{P}_+(\underline{X})$, and to see this extension is continuous we just need to check that the rank doesn't drop. But the image of $(Q(\underline{X})-i\alpha)^{\nu-\beta_-}$ is clearly spanned by the first β_- standard basis vectors of $\mathbb{C}^\nu$, so we are done. □

Remark 7.1. 1. The same argument shows $\mathbb{F}_+(X)=\text{image}\ \mathcal{P}_-(X)$ extends continuously to $\underline{X}$ with $\mathbb{F}_+(\underline{X})$ the span of the first β_+ standard basis vectors of $\mathbb{C}^\nu$.

2. The result of this appendix extends to much more general viscosities by a different argument. See [MZ2].

8. Appendix C: Limits as $z\to\pm\infty$ of slow modes at zero frequency

We've allotted this separate short appendix to the proof of Proposition 3.3 because of its central importance for understanding the connection between viscous and inviscid stability. The result is used in the proof of the Zumbrun-Serre theorem, and also in the propositions of Lecture 3 that remove the translational degeneracy in the SF regime.

Proof of Proposition 3.3. We may write elements of $K_{\hat{H}}(q,\hat{\zeta},\rho)$ as $(w_\pm,0)$, so slow modes can be expressed as

$$U_\pm=Z_\pm Y_\pm\begin{pmatrix}e^{zH_\pm}w_\pm\\0\end{pmatrix}. \tag{8.1}$$

Recalling properties of $Z_\pm(q,z,\zeta)$ (2.38) and $Y_\pm(q,\zeta)$ (2.44) we compute

$$\lim_{z\to\pm\infty}Z_\pm(q,z,0)Y_\pm(q,0)=\begin{pmatrix}I&0\\0&I\end{pmatrix}\begin{pmatrix}I&\mathcal{A}^{-1}_{\pm\infty}(q,0)\\0&I\end{pmatrix}. \tag{8.2}$$

Since $H_\pm(q,0)=0$ we obtain

$$\lim_{z\to\pm\infty} U_\pm(q,z,\hat\zeta,0)=\begin{pmatrix} w_\pm(q,\hat\zeta,0)\\ 0\end{pmatrix}\in K_{\hat H}(q,\hat\zeta,0.) \tag{8.3}$$

□

9. Appendix D: Evans ⇒ transversality + uniform stability

In this section we prove the Zumbrun-Serre result, Theorem 3.1. We'll make use of the discussion of slow and fast modes in Lecture 3.

The curvature of the inviscid shock plays no role here, so it's enough to consider an $m\times m$ system of viscous conservation laws

$$u_t+f_1(u)_x+f_2(u)_y-\triangle u=u_t+A_1(u)u_x+A_2(u)u_y-\triangle u=0, \tag{9.1}$$

a planar inviscid shock ($x=0,U_\pm$), and a stationary solution $W(x)$ (the profile) of (9.1) satisfying the integrated profile equation

$$\begin{aligned}&W'=f_1(W)-f_1(U_-)\text{ and}\\&W(x)\to U_\pm\text{ as }x\to\pm\infty.\end{aligned} \tag{9.2}$$

Linearize (9.1) about W, Fourier-Laplace transform in t, and Fourier transform in y to get the eigenvalue equation

$$(i\tau+\gamma)w+(A_1(W)w)'+i\eta A_2(W)w-w''+\eta^2 w=0. \tag{9.3}$$

Let $\zeta=(\tau,\gamma,\eta)=\rho\hat\zeta$, where $\gamma\geq 0$, $0\leq\rho\leq\rho_0$. For $\rho>0$ let $\{w_1^\pm(x,\hat\zeta,\rho),\dots,w_m^\pm\}$ be a basis for the decaying solutions of (9.3) in $\pm x\geq 0$. By the result of Appendix B these functions can be chosen to be smooth in $\rho>0$ with continuous extensions to $\rho=0$. The Evans function is the wronskian

$$D(\hat\zeta,\rho)=\det\begin{pmatrix} w_+^1 & \dots & w_+^m & w_-^1 & \dots & w_-^m\\ w_+^{1'} & \dots & w_+^{m'} & w_-^{1'} & \dots & w_-^{m'}\end{pmatrix}|_{x=0}. \tag{9.4}$$

Note that when $\rho=0$ equation (9.3) becomes the linearized profile equation

$$w''-(A_1(W)w)'=0. \tag{9.5}$$

Since W' satisfies (9.5) on the whole line, we have $D(\hat\zeta,0)=0$. We want to show

$$D(\hat\zeta,\rho)=\rho\beta\Delta(\hat\zeta)+o(\rho)\text{ as }\rho\to 0, \tag{9.6}$$

where β is a transversality constant defined below and $\Delta(\hat\zeta)$ is the uniform stability determinant (6.5).

Suppose $A_1(U_+)$ has k positive eigenvalues and $A_1(U_-)$ has l negative eigenvalues. The inviscid shock is a Lax shock, so $k+l=m-1$, and hence $(m-k)+(m-l)=m+1$. There are $m-k$ exponentially decaying solutions of (9.5) in $x\geq 0$, and $m-l$ in $x\leq 0$. Calling these $w_+^1(x),\dots,w_+^{m-k}$ and $w_-^1(x),\dots,w_-^{m-l}$ respectively, we may arrange so that the similarly labeled elements $w_\pm^j(x,\hat\zeta,\rho)$ in the bases chosen above are smooth extensions of the $w_\pm^j(x)$ to small nonzero frequencies; so $w_\pm^j(x,\hat\zeta,0)=w_\pm^j(x)$ (see (10.5) for this kind of extension). These extensions decay exponentially to zero in x for $\rho\geq 0$. We call these the *fast* modes. The vectors given by the $w_\pm^j(0)$ (a total of

$m+1$ vectors) span the tangent spaces to the stable/unstable manifolds of (9.2) at $W(0)$ for the rest points $U_\pm$. Moreover, we may suppose the $w^j_\pm$ are chosen so that $w^1_+(x) = w^1_-(x) = W'(x)$. The connection is *transversal* $\Leftrightarrow$ the $m \times m$ determinant

$$\det(w^1_+(0), \dots, w^{m-k}_+(0), w^2_-(0), \dots, w^{m-l}_-(0)) = \beta \tag{9.7}$$

is not zero.

Note that the limiting versions of (9.5), namely

$$w'' - A_1(U_\pm)w' = 0 \text{ in } \pm x \geq 0 \tag{9.8}$$

also have constant solutions. The remaining $k + l = m - 1$ basis elements used in defining D (called the *slow* modes)

$$w^j_\pm(x, \hat{\zeta}, \rho), j = m-k+1, \dots, m \ (+case);\ j = m-l+1, \dots, m \ (-case) \tag{9.9}$$

have the property that $w^j_\pm(x, \hat{\zeta}, 0)$ decay to nonzero constant vectors as $x \to \pm\infty$. Recalling (3.31) we see that we can choose the slow modes so that

$$\lim_{x\to\pm\infty} w^j_\pm(x, \hat{\zeta}, 0) = r^j_\pm(\hat{\zeta}), \tag{9.10}$$

where the $r^j_\pm(\hat{\zeta})$ are the vectors appearing in the definition of $\Delta(\hat{\zeta})$ (6.5) with indices relabeled.

Now we begin the computation of $D(\hat{\zeta}, \rho)$. Set $z_\pm(x) = \partial_\rho w^1_\pm(x, \hat{\zeta}, 0)$. Using the special property of $w^1_\pm$ we may write

$$\begin{pmatrix} w^1_+ \\ w^{1'}_+ \end{pmatrix} - \begin{pmatrix} w^1_- \\ w^{1'}_- \end{pmatrix} = \rho \begin{pmatrix} z_+ - z_- \\ z'_+ - z'_- \end{pmatrix} + o(\rho). \tag{9.11}$$

Use the column operation given by the left side of (9.11) to replace the w^1_- column in (9.4) by the right side of (9.11).

Since (9.5) implies that $w' - A_1(W)w$ is constant, this suggests using the row operation $w' - A_1 w$ to simplify the determinant, provided we can identify the constants.

Integrate (9.5) from $\pm\infty$ to x in $\pm x \geq 0$ to get

$$w' - A_1(W)w = -(A_1(W)w)|_{x=\pm\infty} = -A_1(U_\pm)w(\pm\infty, \hat{\zeta}, 0). \tag{9.12}$$

There are three cases. First, for fast modes the right side of (9.12) is clearly 0.

Second, by (9.10) for slow modes $w^j_\pm$ the right side of (9.12) is $-A_1(U_\pm)r^j_\pm(\hat{\zeta})$ for $r^j_\pm$ as above.

The third case is $z_+ - z_-$. Recall that $w^1_\pm$ satisfy (9.3) on $\pm x \geq 0$. Write the frequencies in (9.3) in polar coordinates, substitute in $w^1_\pm(x, \hat{\zeta}, \rho)$, differentiate with respect to ρ, and evaluate at $\rho = 0$ to get

$$(i\hat{\tau} + \hat{\gamma})W' + (A_1(W)z_\pm)' + i\hat{\eta}A_2(W)W' - z''_\pm = 0 \tag{9.13}$$

(recall $w^1_\pm(x, \hat{\zeta}, 0) = W'(x)$. Integrate from $\pm\infty$ to x in $\pm x \geq 0$ to get

$$(i\hat{\tau} + \hat{\gamma})W + A_1(W)z_\pm + i\hat{\eta}f_2(W) - z'_\pm - \{(i\hat{\tau} + \hat{\gamma})U_\pm + i\hat{\eta}f_2(U_\pm)\} = 0 \tag{9.14}$$

(we used $z_\pm(\pm\infty,\hat{\zeta},0)=0$). Finally, subtract the $(-)$ equation from the $(+)$ equation to get

$$(z_- - z_+)' - A_1(z_- - z_+) = (i\hat{\tau}+\hat{\gamma})[U] + i\hat{\eta}[f(U)]. \tag{9.15}$$

So in cases 1,2,3 the row operation $w' - A_1 w$ produces the results 0, $-A_1(U_\pm)r^j_\pm(\hat{\zeta})$, and $(i\hat{\tau}+\hat{\gamma})[U]+i\hat{\eta}[f(U)]$ respectively. Apply the row operation to get (up to a sign)

(9.16)
$$\rho \det \begin{pmatrix} w^j_\pm(0)\ (m\ fast) & * & * \\ 0 & A_1(U_\pm)r^j_\pm(\hat{\zeta})\ (m-1\ slow) & (i\hat{\tau}+\hat{\gamma})[U]+i\hat{\eta}[f(U)] \end{pmatrix} + o(\rho).$$

The upper left $m\times m$ determinant is β, the transversality constant. The lower right $m\times m$ determinant is $\Delta(\hat{\zeta})$, the Majda uniform stability determinant, so up to a sign we have (9.6). We can redefine β to correct the sign if necessary.

10. Appendix E: Proofs in Lecture 3

10.1. **Construction of R.** To construct R as in (3.39) we must find (for ρ small) an exponentially decaying function, vanishing at $\rho=0$ which satisfies

$$\begin{aligned} &R_z - GR = \mathcal{B} \text{ in } \pm z \geq 0,\ l(q)\cdot r_\pm = -p(\zeta) \text{ where} \\ &\mathcal{B} = (B^0)^{-1}\left(0, (i\tau+\gamma+\eta^2)W_z + i\eta g'(W)W_z + 2i\eta\psi^0_y W_{zz}\right). \end{aligned} \tag{10.1}$$

First find an exponentially decaying function $R^1=(r^1,r^2)$ such that $R_z - GR^1 = \mathcal{B}$ with

$$R^1 = (B^0)^{-1}\left((i\tau+\gamma+\eta^2)R^{11} + i\eta R^{12} + 2i\eta\psi^0_y R^{13}\right), \tag{10.2}$$

where R^{1j}, $j=1,2,3$ satisfy $R^{1j}_z - GR^{1j} = F$, with

$$F = (0,W_z), (0,g'(W)W_z), (0,W_{zz}) \tag{10.3}$$

respectively. This is easy after MZ conjugation which replaces G by $G_{\pm\infty}$, whose spectrum is described at the beginning of Lecture 3.

Next we must add a correction to arrange the boundary condition. Recall that $\mathcal{P}(q,z)=(W_z,W_{zz})$ is a fast decaying solution of $\mathcal{P}_z - G(q,z,0)\mathcal{P}=0$. Using the correspondence with $G_{HP\pm}$ form, this means that

$$\mathcal{P}(q,z) = Z_\pm(q,z,0)Y_\pm(q,0)\begin{pmatrix} 0 \\ e^{zP_\pm(q,0)}c_\pm(q)\end{pmatrix}, \tag{10.4}$$

for some $c_\pm(q)$ in the negative (resp., positive) eigenspace of $P_\pm(q,0)$. Thus, we can construct smooth extensions to nonzero frequency $\mathcal{P}_\pm(q,z,\zeta)$ satisfying $\mathcal{P}_z - G\mathcal{P}=0$ in $\pm z\geq 0$ by taking

$$\mathcal{P}(q,z,\zeta) = Z_\pm(q,z,\zeta)Y_\pm(q,\zeta)\begin{pmatrix} 0 \\ e^{zP_\pm(q,0)}\pi_\pm(q,\zeta)c_\pm(q)\end{pmatrix} \tag{10.5}$$

where $\pi_\pm(q,\zeta)$ projects onto the negative (resp., positive) eigenspace of $P_\pm$. Writing $\mathcal{P}=(p^1,p^2)$ we have

$$l(q)\cdot p^1_\pm(q,0,\zeta) = d_\pm(q,\zeta) \sim 1 \text{ for } \rho \text{ small}. \tag{10.6}$$

So if we define

$$R = R^1 - \alpha(q,\zeta)\mathcal{P}, \text{ where}$$
$$\alpha_\pm = \left(l(q)\cdot r^1_\pm(q,0,\zeta) + p(\zeta)\right)\cdot \frac{1}{d_\pm(q,\zeta)}, \tag{10.7}$$

then R has all the required properties.

10.2. **Propositions 3.4 and 3.5.**

Proof of Proposition 3.4. From (3.39) we obtain

$$\begin{aligned}&\partial_z\hat{R} - G(q,z,0)\hat{R} = \hat{\mathcal{B}} \text{ or equivalently,}\\ &L_u(q,z,\hat{\zeta},0,\partial_z)\hat{r} = -L_\psi(q,z,\hat{\zeta},0) = (i\hat{\tau}+\hat{\gamma})W_z + i\hat{\eta}g'(W)W_z + i\hat{\eta}2\psi^0_y W_{zz}.\end{aligned} \tag{10.8}$$

In turn this is the same as

$$-B^0\partial_z^2\hat{r}_\pm + \partial_z(A_\nu(W,d\psi^0)\hat{r}) = \partial_z\hat{\beta}, \tag{10.9}$$

where $\hat{\beta}$ is a primitive of $-L_\psi$. Integrating $\int_{\pm\infty}^z$ in $\pm z \geq 0$ gives

$$-B^0\partial_z\hat{r}_\pm + A_\nu\hat{r}_\pm = \hat{\beta}(z) - \left((i\hat{\tau}+\hat{\gamma})U^0_\pm + i\hat{\eta}g(U^0_\pm)\right). \tag{10.10}$$

Finally, subtract the $+$ equation from the $-$ equation and evaluate at $z=0$ to find

$$[B^0\partial_z\hat{r} - A_\nu\hat{r}] = (i\hat{\tau}+\hat{\gamma})[U^0] + i\hat{\eta}[g(U^0)] \neq 0, \tag{10.11}$$

since the uniform stability determinant $\Delta(q,\hat{\zeta}) \neq 0$ (see (6.5)). The Proposition follows immediately from (10.11).

□

Proof of Proposition 3.5. Suppose the intersection

$$\ker\tilde{\Gamma}\cap(E_+(q,\hat{\zeta},0)\times E_-(q,\hat{\zeta},0))$$

is nontrivial. Then there exist $U_\pm(q,z,\hat{\zeta},0)$ satisfying $U_z - G(q,\zeta,0)U = 0$ in $\pm z \geq 0$ with initial data in $E_+(q,\hat{\zeta},0)\times E_-(q,\hat{\zeta},0)$ such that (with $U=(u,u_z)$)

$$[U](q,0,\hat{\zeta},0) = c[\hat{R}](q,0,\hat{\zeta},0) \text{ for some } c \text{ and } l(q)\cdot u_+ = 0. \tag{10.12}$$

In Remark 3.2 we ruled out the possibility $c=0$. So suppose $c\neq 0$ and define $\mathcal{U} = (\mu,\mu_z)$ by

$$\mathcal{U}_\pm = c\hat{R}_\pm(q,z,\zeta,0) - U_\pm(q,z,\hat{\zeta},0). \tag{10.13}$$

Then

$$\partial_z\mathcal{U} - G(q,z,0)\mathcal{U} = c\hat{\mathcal{B}}(q,0,\hat{\zeta},0) \text{ and } [\mathcal{U}] = 0 \tag{10.14}$$

or equivalently

(10.15)
$$\begin{aligned}&L_u(q,z,\hat{\zeta},0,\partial_z)\mu_\pm = -cL_\psi(q,z,\hat{\zeta},0) = c\left((i\hat{\tau}+\hat{\gamma})W_z + i\hat{\eta}g'(W)W_z + i\hat{\eta}2\psi^0_y W_{zz}\right),\\ &[\mu] = [\mu_z] = 0.\end{aligned}$$

Note that by (8.3)

$$\begin{aligned}\lim_{z\to\pm\infty}\mu_\pm(q,z,\hat\zeta,0) &= -\lim_{z\to\pm\infty}u_\pm\in\operatorname{span}\{r^j_\pm(q,\hat\zeta)\}\\ \lim_{z\to\pm\infty}\partial_z\mu_\pm &= 0\end{aligned}\tag{10.16}$$

for $r^j_\pm(q,\hat\zeta)$ as in (6.5) (or (3.31)).

Next integrate (10.15) $\int_{\pm\infty}^z$ in $\pm z\geq 0$ to get with β as in (10.9)

$$-B^0\partial_z\mu_\pm + A_\nu\mu_\pm - A_\nu\mu_\pm(\pm\infty) = c\hat\beta(z) - c\left((i\hat\tau+\gamma)U^0_\pm + i\hat\eta g(U^0_\pm)\right).\tag{10.17}$$

Because of (10.16), if we set $z=0$ and subtract the $+$ equation from the $-$ equation, we can find a nontrivial linear combination (since $c\neq 0$) of the vectors

$$A_\nu r^j_\pm(q,\hat\zeta)\text{ and }(i\hat\tau+\hat\gamma)[U^0]+i\hat\eta[g(U^0)]\tag{10.18}$$

that vanishes. This contradicts uniform stability of the inviscid shock.

□

11. Appendix F: The HF estimate

The first step is to understand the spectrum of $\mathcal{G}(q,z,\zeta)$ for large $|\zeta|$. We'll set

$$\langle\zeta\rangle = (\tau^2+\gamma^2+\eta^4)^{1/4},\tag{11.1}$$

reflecting the parabolic quasihomogeneity where τ and γ have weight two, and η has weight one.

Proposition 11.1. *For C large enough and $|\zeta|\geq C$, $\mathcal{G}(q,z,\zeta)$ has $2m$ eigenvalues in $\Re\mu>0$ and $2m$ eigenvalues in $\Re\mu<0$ with*

$$|\Re\mu| > C\langle\zeta\rangle.\tag{11.2}$$

Proof. Consider the block

$$G_+(q,z,\zeta) = \begin{pmatrix}0 & I\\ \mathcal{M} & \mathcal{A}\end{pmatrix}\tag{11.3}$$

(recall (2.32)).

Define

$$\lambda=\frac{1}{\langle\zeta\rangle},\ \check\zeta=(\check\tau,\check\gamma,\check\eta)=\left(\frac{\tau}{\langle\zeta\rangle^2},\frac{\gamma}{\langle\zeta\rangle^2},\frac{\eta}{\langle\zeta\rangle}\right),\tag{11.4}$$

and note that $\langle\zeta\rangle=1$, so $\check\zeta\in\check S^2_+$, the parabolic unit half sphere in $\check\gamma\geq 0$. One might call these "parabolic polar coordinates at ∞".

Write

$$\begin{aligned}\mathcal{M}(q,z,\zeta) &= \langle\zeta\rangle^2\check{\mathcal{M}}(q,z,\check\zeta,\lambda)\\ \mathcal{A}(q,z,\zeta) &= \langle\zeta\rangle\check{\mathcal{A}}(q,z,\check\zeta,\lambda),\end{aligned}\tag{11.5}$$

where

$$\begin{aligned}\check M(q,z,\check\zeta,\lambda) &= (B^0)^{-1}\left((i\check\tau+\check\gamma+\check\eta^2)I\right)+O(\lambda)=\check{\mathcal{M}}_0(q,z,\check\zeta)+O(\lambda)\\ \check{\mathcal{A}}(q,z,\check\zeta,\lambda) &= (B^0)^{-1}2\psi^0_y i\check\eta I+O(\lambda)=\check{\mathcal{A}}_0(q,z,\check\zeta)+O(\lambda).\end{aligned}\tag{11.6}$$

We have

(11.7)
$$\begin{pmatrix}\langle\zeta\rangle I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}0 & I\\ \mathcal{M} & \mathcal{A}\end{pmatrix}\begin{pmatrix}\langle\zeta\rangle^{-1}I & 0\\ 0 & I\end{pmatrix}=\langle\zeta\rangle\begin{pmatrix}0 & I\\ \check{\mathcal{M}} & \check{\mathcal{A}}\end{pmatrix}=\langle\zeta\rangle\begin{pmatrix}0 & I\\ \check{\mathcal{M}}_0 & \check{\mathcal{A}}_0\end{pmatrix}+O(1).$$

Since (q,z) dependence enters only through $W(q,z)$, we see that

$$\check{G}_0=\begin{pmatrix}0 & I\\ \check{\mathcal{M}}_0 & \check{\mathcal{A}}_0\end{pmatrix} \tag{11.8}$$

depends on a compact set of parameters. We claim $\check{G}_0$ has no eigenvalues on the imaginary axis. Then, setting $(\check{\tau},\check{\eta})=0$ easily yields a count of m eigenvalues in each of the regions $\pm\Re\mu>0$, thus completing the proof.

To prove the claim, suppose $i\check{\xi}$ is a pure imaginary eigenvalue of $\check{G}_0$ associated to the eigenvector (u,v). A short computation shows

$$\big(B^0\check{\xi}^2+\check{\eta}^2-2\psi^0_y\check{\eta}\check{\xi}+(i\check{\tau}+\check{\gamma})\big)u=0. \tag{11.9}$$

Since

$$B^0\check{\xi}^2+\check{\eta}^2-2\psi^0_y\check{\eta}\check{\xi}\geq C|\check{\xi},\check{\eta}|^2 \tag{11.10}$$

for some $C>0$, we conclude $\check{\gamma}\leq -C|\check{\xi},\check{\eta}|^2$. Hence $|\check{\xi},\check{\eta}|=0$ which implies $|\check{\tau},\check{\gamma}|=0$, and this contradicts $\langle\check{\zeta}\rangle=1$.

□

Remark 11.1. This proof makes precise the sense in which the parabolic part of the operator is "dominant" in the HF regime.

11.1. **Block stucture.** First we rewrite the $2m\times 2m$ transmission problem

$$U_z-G(q,z,\zeta)U=F \text{ in } \pm z\geq 0, [U]=0. \tag{11.11}$$

With $U=(u,v)$ set

$$U_1=(u_1,v_1), \text{ with } u_1=\langle\zeta\rangle u,\ v_1=v. \tag{11.12}$$

Then (11.11) is the same as

$$\partial_z U_1-\langle\zeta\rangle\check{G}(q,z,\check{\zeta},\lambda)U_1=F \text{ in } \pm z\geq 0, [U_1]=0, \tag{11.13}$$

where

$$\check{G}=\begin{pmatrix}0 & I\\ \check{\mathcal{M}} & \check{\mathcal{A}}\end{pmatrix}. \tag{11.14}$$

Next, with doubling notation as in (3.17) we rewrite the problem as the $4m\times 4m$ system on $z\geq 0$

$$\partial_z U_1-\langle\zeta\rangle\check{\mathcal{G}}U_1=\mathcal{F},\ \Gamma U_1=0 \tag{11.15}$$

where

$$\check{\mathcal{G}}(q,z,\check{\zeta},\lambda)=\begin{pmatrix}\check{G}(z) & 0\\ 0 & -\check{G}(-z)\end{pmatrix}. \tag{11.16}$$

The spectral separation proved in Proposition 11.1 implies for $|\zeta|$ large that there exists a smooth conjugator $T(q,z,\check\zeta,\lambda)$ such that

$$T^{-1}\check{\mathcal{G}}T = \begin{pmatrix} P_g(q,z,\check\zeta,\lambda) & 0 \\ 0 & P_d \end{pmatrix} := \check{\mathcal{G}}_{gd}, \tag{11.17}$$

where $\Re P_g > CI$, $\Re P_d < -CI$ with the spectrum of P_g, P_d contained in a compact subset of $\pm\Re\mu > 0$ respectively.

11.2. **Symmetrizer and estimate.** Setting $U_1 = TU_2$ and noting that T has z dependence, we reduce (11.15) to

$$\begin{aligned} &\partial_z U_2 - \langle\zeta\rangle\check{\mathcal{G}}_{gd}U_2 = T^{-1}\mathcal{F} - T^{-1}T_zU_2 := \mathcal{F}', \\ &\Gamma TU_2 := \Gamma'(q,\check\zeta,\lambda)U_2 = 0. \end{aligned} \tag{11.18}$$

Let $\mathbb{F}^\infty(q,\check\zeta,\lambda)$ be the $2m$ dimensional generalized eigenspace of $\check{\mathcal{G}}(q,0,\check\zeta,\lambda)$ corresponding to eigenvalues with negative real part. Clearly,

$$\mathbb{F}^\infty(q,\check\zeta,\lambda) = T(q,0,\check\zeta,\lambda)\mathbb{F} \tag{11.19}$$

where $\mathbb{F} = \{(0,a) : a \in \mathbb{C}^{2m}\}$ as before (see (4.8)), and

$$\ker\Gamma'(q,\check\zeta,\lambda)\cap\mathbb{F} = \{0\} \Leftrightarrow \ker\Gamma\cap\mathbb{F}^\infty(q,\check\zeta,\lambda) = \{0\}. \tag{11.20}$$

In view of Proposition 11.1 and (11.16), the second equality in (11.20) holds for λ small, since a vector w is in the negative invariant space of $\check G(0)$ if and only if it is in the positive invariant space of $-\check G(0)$.

We can now finish by arguing just as in the MF case (4.10)-(4.20). In view of the block structure of (11.18) we set $U = U_g + U_d$ as before and take S of the form

$$S = \begin{pmatrix} cI & 0 \\ 0 & -I \end{pmatrix}. \tag{11.21}$$

For c large enough we have

$$\begin{aligned} &\Re(S\check{\mathcal{G}}_{gd}) \geq I \text{ in } z \geq 0 \\ &S + C(\Gamma')^*\Gamma' \geq I \text{ on } z = 0 \end{aligned} \tag{11.22}$$

for some $C > 0$. This implies

$$\begin{aligned} &\langle\zeta\rangle|U_2|_2^2 + |U_2|^2 \leq C\frac{|\mathcal{F}'|_2^2}{\langle\zeta\rangle}, \text{ or} \\ &\langle\zeta\rangle|U_2|_2 + \langle\zeta\rangle^{1/2}|U_2| \leq C|\mathcal{F}'|_2. \end{aligned} \tag{11.23}$$

Recall the definition of $\mathcal{F}'$, take $|\zeta|$ large to absorb the $T^{-1}T_zU_2$ term, and replace U_2 by U_1 to get

$$\langle\zeta\rangle|U_1|_2 + \langle\zeta\rangle^{1/2}|U_1| \leq C|\mathcal{F}|_2. \tag{11.24}$$

Since $u_1 = \langle\zeta\rangle u$, $v_1 = v$ for $U = (u,v)$ as in (11.11), when the forcing F has the form $F = (0,f)$ we obtain

$$\langle\zeta\rangle^2|u|_2 + \langle\zeta\rangle|u_z|_2 + \langle\zeta\rangle^{3/2}|u| + \langle\zeta\rangle^{1/2}|u_z| \leq C|f|_2. \tag{11.25}$$

This finishes the proof of the estimate (4.59)(c) quoted in Lecture 4.

Remark 11.2. Note that we have not used the Evans assumption in HF at all. Instead we were able to use the positive definiteness of the viscosity to deduce that the Evans condition (11.20) holds uniformly for large $|\zeta|$.

12. Appendix G: Transition to PDE estimates

In this appendix we try to give a brief indication of how the same matrix symbols we've constructed here in the course of proving uniform estimates for systems of ODEs depending on frequencies as parameters can be used to prove estimates for the original linearized system of PDEs. For details we refer to the appendix of [GMWZ2].

For purposes of illustration it is simplest to work with smooth matrix symbols $a(q,\zeta)$ supported in MF, the midfrequency region. First, remember that ζ here is really $\tilde{\zeta}=\epsilon\zeta=\epsilon(\tau,\gamma,\eta)$ (recall (2.23)). Starting with $a(q,\tilde{\zeta})$ we unfreeze $q=(t,y)$ and replace $\tilde{\zeta}$ by $\epsilon\zeta$:

$$a(q,\tilde{\zeta})\to a(q,\epsilon\zeta)\to a(t,y,\epsilon\zeta). \tag{12.1}$$

Next we associate a semiclassical pseudodifferential operator a_D to $a(t,y,\epsilon\zeta)$ whose action on a function $u(t,y)$ is given by

$$(a_Du)(t,y)=\int e^{it\tau+iy\eta}a(t,y,\epsilon\tau,\epsilon\gamma,\epsilon\eta)\hat{u}(\tau,\eta)d\tau d\eta. \tag{12.2}$$

Note that the Fourier inversion formula implies that semiclassical *differential* operators are special cases of the operators defined by (12.2).

It is not hard using basic properties of the Fourier transform to show that

$$a_D:L^2\to L^2. \tag{12.3}$$

If $b(q,\tilde{\zeta})$ is another such symbol, consider the composite operator a_Db_D. It is not immediately obvious that the composition is an operator of the same type. Ideally, one would have a relationship like $a_Db_D=(ab)_D$, where ab is the ordinary matrix product of the symbols a and b. Instead, one has

$$a_Db_D=(ab)_D+\epsilon R_D,\ \text{ where } R_D:L^2\to L^2. \tag{12.4}$$

In other words the ideal relationship does hold, modulo an error with L^2 norm that can be taken arbitrarily small. Such errors are often negligible in energy estimates proved using the pseudodifferential calculus.

There is a similar relationship for adjoints

$$(a_D)^*=(a^*)_D+\epsilon R_D \tag{12.5}$$

for R_D as above. The main point here is that pseudodifferential operators behave just like their symbols under the operations of composition and taking adjoints, except for errors that are often negligible.

Suppose next that we have symbols $S(q,\tilde{\zeta})$, $G(q,\tilde{\zeta})$, and $\chi(\tilde{\zeta})$ such that

$$\text{Re }S(q,\tilde{\zeta})G(q,\tilde{\zeta})\geq CI \text{ for } \tilde{\zeta}\in\text{supp }\chi \tag{12.6}$$

(recall for example the MF estimate in Lecture 4). Then *Garding's inequality*, which can easily be proved using the calculus outlined above, implies

$$\text{Re } (S_D G_D \chi_D u, \chi_D u) \geq C(|\chi_D u|^2_{L^2} - \epsilon^2 |u|^2_{L^2}). \tag{12.7}$$

Of course, this is the PDE analogue of the estimate on $\Re(S\mathcal{G}_{gd}U, U)$ that we used in (4.17), (4.18).

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