The Mach stem equation and amplification in strongly nonlinear geometric optics

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Abstract

We study highly oscillating solutions to a class of weakly well-posed hyperbolic initial boundary value problems. Weak well-posedness is associated with an amplification phenomenon of oscillating waves on the boundary. In the previous works [CGW14, CW14], we have rigorously justified a *weakly nonlinear* regime for *semilinear* problems. In that case, the forcing term on the boundary has amplitude $O(\varepsilon^2)$ and oscillates at a frequency $O(1/\varepsilon)$. The corresponding exact solution, which has been shown to exist on a time interval that is independent of $\varepsilon \in (0, 1]$, has amplitude $O(\varepsilon)$. In this paper, we deal with the exact same scaling, namely $O(\varepsilon^2)$ forcing term on the boundary and $O(\varepsilon)$ solution, for *quasilinear* problems. In analogy with [CGM03], this corresponds to a *strongly nonlinear* regime, and our main result proves solvability for the corresponding WKB cascade of equations, which yields existence of approximate solutions on a time interval that is independent of $\varepsilon \in (0, 1]$. Existence of exact solutions close to approximate ones is a stability issue which, as shown in [CGM03], highly depends on the hyperbolic system and on the boundary conditions; we do not address that question here.

This work encompasses previous formal expansions in the case of weakly stable shock waves [MR83] and two-dimensional compressible vortex sheets [AM87]. In particular, we prove well-posedness for the leading amplitude equation (the "Mach stem equation") of [MR83] and generalize its derivation to a large class of hyperbolic boundary value problems and to periodic forcing terms. The latter case is solved under a crucial nonresonant assumption and a small divisors condition.

Contents

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1 Introduction

1.1 General presentation

This article is devoted to the analysis of high frequency solutions to quasilinear hyperbolic initial boundary value problems. Up to now, the rigorous construction of such solutions is known in only a few situations and highly depends on the well-posedness properties of the boundary value problem one considers. In the case where the so-called *uniform Kreiss-Lopatinskii condition* is satisfied, the existence of *exact* oscillating solutions on a fixed time interval has been proved by one of the authors in [Wil02], see also [Wil96] for semilinear problems. The asymptotic behavior of exact solutions as the wavelength tends to zero is described in [CGW11] for wavetrains and in [CW13] for pulses. The main difference between the two problems is that in the wavetrains case, resonances can occur between a combination of three phases, giving rise to integro-differential terms in the equation that governs the leading amplitude of the solution[1]. Resonances do not occur at the leading order[2] for pulses, which makes the leading amplitude equation easier to deal with in that case.

In this article, we pursue our study of *weakly well-posed* problems and consider situations where the uniform Kreiss-Lopatinskii condition breaks down. Let us recall that in that case, high frequency oscillations can be amplified when reflected on the boundary. As far as we know, this phenomenon was first identified by Majda and his collaborators, see for instance [MR83, AM87, MA88] in connection with the formation of specific wave patterns in compressible fluid dynamics. Asymptotic expansions in the spirit of [AM87] are also performed in the recent work [WY14]. In various situations (depending on the scaling of the source terms and on the number of phases), these authors managed to derive an equation that governs the leading amplitude of the solution. Solving the leading amplitude equation in [MR83] and constructing exact and/or approximate oscillating solutions was left open. As far as we know, the rigorous justification of such expansions has not been considered in the literature so far. The present article follows previous works where we have given a rigorous justification of the amplification phenomenon: first for *linear* problems in [CG10], and then for *semilinear* problems in [CGW14, CW14]. These previous works considered either linear problems, or a weakly nonlinear regime of oscillating solutions for which the existence and asymptotic behavior of *exact* oscillating solutions can be studied on a fixed time interval.

The regime considered in [MR83, AM87], and that we shall also consider in this article, goes beyond the one considered in [CGW14, CW14]. In analogy with [CGM03], this regime will be referred to as that of *strong oscillations*. We extend the analysis of [MR83, AM87] to a general framework, not restricted to the system of gas dynamics, and explain why the problem of vortex sheets considered in [AM87] and the analogous one in [WY14] yield a much simpler equation than the problem of shock waves in [MR83]. We also clarify the causality arguments used in [MR83, AM87] to discard some of the terms in the (formal) asymptotic expansion of the highly oscillating solution. We need however to make a crucial assumption in order to analyze this asymptotic expansion, namely we need to assume that no resonance occurs between the phases. This is no major concern for pulses because interactions are not visible at the leading order, and this may be the reason why this aspect was not mentioned in [MR83]. Resonances can have far worse consequences when dealing with wavetrains, and what saves the day in [AM87] is that there are too few phases to allow for resonances. This explains why the amplitude equation in [AM87] and the corresponding one in [WY14] reduce to the standard Burgers equation. When the system admits at least

[1] This is not specific to the boundary conditions and is also true for the Cauchy problem in the whole space, see, e.g., [HMR86, JMR95].

[2] Interactions between pulses associated with different phases need to be considered only when dealing with the construction of correctors to the leading amplitude.

three phases (two incoming and one outgoing), and even in the absence of resonances, the amplification phenomenon gives rise, as in [MR83], to integro-differential terms in the equation for the amplitude that determines the trace of the leading profile. We refer to the latter equation as "the Mach stem equation", and show how it arises more generally in weakly stable (WR class) hyperbolic boundary problems with a strongly nonlinear scaling, both in the wavetrain setting, where the equation we derive appears to be completely new, and in the pulse setting, where the equation coincides with the one derived in [MR83].[3]

Our main results establish the well-posedness of the Mach stem equation in both settings, and then use those solvability results to construct *approximate* highly oscillating solutions on a *fixed* time interval to the underlying hyperbolic boundary value problems. In the wavetrain case we are able to construct approximate solutions of arbitrarily high order under a crucial nonresonant condition and a small divisors condition; in the pulse setting we construct approximate solutions up to the point at which further "correctors" are too large to be regarded as correctors.

In Appendix B we compute the formal large period limit of the Mach stem equation for wavetrains, and find a surprising discrepancy (described further below) between that limit and the Mach stem equation for pulses derived in [MR83].

The construction of *exact* oscillating solutions close to approximate ones is a stability issue that is far from obvious in such a strong scaling. We refer to [CGM03] for indications on possible instability issues and postpone the stability problem in our context to a future work. In any case, it is very likely that no general answer can be given and that stability vs instability of the family of approximate solutions will depend on the system and/or on the boundary conditions, see for instance [CGM04] for further results in this direction.

Notation

Throughout this article, we let $\mathcal{M}_{n,N}(\mathbb{K})$ denote the set of $n \times N$ matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and we use the notation $\mathcal{M}_N(\mathbb{K})$ when $n = N$. We let I denote the identity matrix, without mentioning the dimension. The norm of a (column) vector $X \in \mathbb{C}^N$ is $|X| := (X^* X)^{1/2}$, where the row vector X^* denotes the conjugate transpose of X. If X, Y are two vectors in $\mathbb{C}^N$, we let $X \cdot Y$ denote the quantity $\sum_j X_j Y_j$, which coincides with the usual scalar product in $\mathbb{R}^N$ when X and Y are real. We often use Einstein's summation convention in order to make some expressions easier to read.

The letter C always denotes a positive constant that may vary from line to line or within the same line. Dependance of the constant C on various parameters is made precise throughout the text.

1.2 The equations and main assumptions

In the space domain $\mathbb{R}^d_+ := \{x = (y, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \, : \, x_d > 0\}$, we consider the quasilinear evolution problem with oscillating source term:

$$\begin{cases} \partial_t u_\varepsilon + \sum_{j=1}^d A_j(u_\varepsilon)\, \partial_j u_\varepsilon = 0\,, & t \le T\,,\, x \in \mathbb{R}^d_+\,, \\ b(u_\varepsilon|_{x_d=0}) = \varepsilon^2\, G\left(t, y, \dfrac{\varphi_0(t,y)}{\varepsilon}\right)\,, & t \le T\,,\, y \in \mathbb{R}^{d-1}\,, \\ u_\varepsilon, G|_{t<0} = 0\,, & \end{cases} \tag{1.1}$$

where the A_j's belong to $\mathcal{M}_N(\mathbb{R})$ and depend in a $\mathcal{C}^\infty$ way on u in a neighborhood of 0 in $\mathbb{R}^N$, b is a $\mathcal{C}^\infty$ mapping from a neighborhood of 0 in $\mathbb{R}^N$ to $\mathbb{R}^p$ (the integer p is made precise below), and the source

[3] The WR class is described in Assumption 1.6.

term G is valued in $\mathbb{R}^p$. It is also assumed that $b(0) = 0$, so that the solution starts from the rest state 0 in negative times and is ignited by the small oscillating source term $\varepsilon^2 G$ on the boundary in positive times. The two main underlying questions of nonlinear geometric optics are:

1. Proving existence of solutions to (1.1) on a fixed time interval (the time $T > 0$ should be independent of the wavelength $\varepsilon \in (0, 1]$).

2. Studying the asymptotic behavior of the sequence u_ε as ε tends to zero. If we let u_ε^{app} denote an approximate solution on $[0, T']$, $T' \leq T$, constructed by the methods of nonlinear geometric optics (that is, solving eikonal equations for phases and suitable transport equations for profiles), how well does u_ε^{app} approximate u_ε for ε small? For example, is it true that
$$\lim_{\varepsilon \to 0} \, |u_\varepsilon - u_\varepsilon^{app}|_{L^\infty([0,T'] \times \mathbb{R}^d_+)} \to 0\,?$$

The above questions are dealt with in a different way according to the functional setting chosen for the source term G in (1.1). More precisely, we distinguish between:

- Wavetrains, for which G is a function defined on $(-\infty, T_0] \times \mathbb{R}^{d-1} \times \mathbb{R}$ that is Θ-periodic with respect to its last argument (denoted θ_0 later on).
- Pulses, for which G is a function defined on $(-\infty, T_0] \times \mathbb{R}^{d-1} \times \mathbb{R}$ that has at least polynomial decay at infinity with respect to its last argument.

The answer to the above two questions highly depends on the well-posedness of the linearized system at the origin:

$$\begin{cases} \partial_t v + \sum_{j=1}^d A_j(0)\, \partial_j v = f\,, & t \leq T\,,\, x \in \mathbb{R}^d_+\,, \\ \mathrm{d}b(0) \cdot v|_{x_d=0} = g\,, & t \leq T\,,\, y \in \mathbb{R}^{d-1}\,, \\ v, f, g|_{t<0} = 0\,. \end{cases} \tag{1.2}$$

The first main assumption for the linearized problem (1.2) deals with hyperbolicity.

Assumption 1.1 (Hyperbolicity with constant multiplicity)**.** *There exist an integer $q \geq 1$, some real functions $\lambda_1, \dots, \lambda_q$ that are analytic on $\mathbb{R}^d \setminus \{0\}$ and homogeneous of degree 1, and there exist some positive integers $\nu_1, \dots, \nu_q$ such that:*

$$\forall\, \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \setminus \{0\}\,, \quad \det \Big[\tau\, I + \sum_{j=1}^d \xi_j\, A_j(0) \Big] = \prod_{k=1}^q \big(\tau + \lambda_k(\xi)\big)^{\nu_k}\,.$$

Moreover the eigenvalues $\lambda_1(\xi), \dots, \lambda_q(\xi)$ are semi-simple (their algebraic multiplicity equals their geometric multiplicity) and satisfy $\lambda_1(\xi) < \cdots < \lambda_q(\xi)$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

For reasons that will be fully explained in Sections 2 and 4, we make a technical complementary assumption.

Assumption 1.2 (Strict hyperbolicity or conservative structure)**.** *In Assumption 1.1, either all integers $\nu_1, \dots, \nu_q$ equal 1 (which means that the operator $\partial_t + \sum_j A_j(0)\, \partial_j$ is strictly hyperbolic), or $A_1(u), \dots, A_d(u)$ are Jacobian matrices of some flux functions $f_1, \dots, f_d$ that depend in a $\mathcal{C}^\infty$ way on u in a neighborhood of 0 in $\mathbb{R}^N$.*

For simplicity, we restrict our analysis to noncharacteristic boundaries and therefore make the following:

Assumption 1.3 (Noncharacteristic boundary)**.** *The matrix $A_d(0)$ is invertible and the Jacobian matrix $B := \mathrm{d}b(0)$ has maximal rank, its rank p being equal to the number of positive eigenvalues of $A_d(0)$ (counted with their multiplicity). Moreover, the integer p satisfies $1 \le p \le N-1$.*

Energy estimates for solutions to (1.2) are based on the normal modes analysis, see [BGS07, chapter 4]. We let $\tau - i\,\gamma \in \mathbb{C}$ and $\eta \in \mathbb{R}^{d-1}$ denote the dual variables of t and y in the Laplace and Fourier transform, and we introduce the symbol

$$\mathcal{A}(\zeta) := -i\,A_d(0)^{-1}\left((\tau - i\gamma)\,I + \sum_{j=1}^{d-1} \eta_j\,A_j(0)\right)\,,\quad \zeta := (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1}\,.$$

For future use, we also define the following sets of frequencies:

$$\Xi := \Big\{(\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus (0,0) \,:\, \gamma \ge 0\Big\}\,, \qquad \Sigma := \Big\{\zeta \in \Xi \,:\, \tau^2 + \gamma^2 + |\eta|^2 = 1\Big\}\,,$$
$$\Xi_0 := \Big\{(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} \setminus (0,0)\Big\} = \Xi \cap \{\gamma = 0\}\,, \qquad \Sigma_0 := \Sigma \cap \Xi_0\,.$$

Two key objects in our analysis are the hyperbolic region and the glancing set that are defined as follows.

Definition 1.4.
- *The hyperbolic region $\mathcal{H}$ is the set of all $(\tau, \eta) \in \Xi_0$ such that the matrix $\mathcal{A}(\tau, \eta)$ is diagonalizable with purely imaginary eigenvalues.*
- *Let $\mathbf{G}$ denote the set of all $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d$ such that $\xi \neq 0$ and there exists an integer $k \in \{1, \dots, q\}$ satisfying*
$$\tau + \lambda_k(\xi) = \frac{\partial \lambda_k}{\partial \xi_d}(\xi) = 0\,.$$
If $\pi(\mathbf{G})$ denotes the projection of $\mathbf{G}$ on the first d coordinates (that is, $\pi(\tau, \xi) := (\tau, \xi_1, \dots, \xi_{d-1})$ for all (τ, ξ)), the glancing set $\mathcal{G}$ is $\mathcal{G} := \pi(\mathbf{G}) \subset \Xi_0$.

We recall the following result that is due to Kreiss [Kre70] in the strictly hyperbolic case (when all integers ν_j in Assumption 1.1 equal 1) and to Métivier [Mét00] in our more general framework.

Theorem 1.5 ([Kre70, Mét00])**.** *Let Assumptions 1.1 and 1.3 be satisfied. Then for all $\zeta \in \Xi \setminus \Xi_0$, the matrix $\mathcal{A}(\zeta)$ has no purely imaginary eigenvalue and its stable subspace $\mathbb{E}^s(\zeta)$ has dimension p. Furthermore, $\mathbb{E}^s$ defines an analytic vector bundle over $\Xi \setminus \Xi_0$ that can be extended as a continuous vector bundle over Ξ.*

For all $(\tau, \eta) \in \Xi_0$, we let $\mathbb{E}^s(\tau, \eta)$ denote the continuous extension of $\mathbb{E}^s$ to the point (τ, η). Away from the glancing set $\mathcal{G} \subset \Xi_0$, $\mathbb{E}^s(\zeta)$ depends analytically on ζ, see [Mét00]. In particular, it follows from the analysis in [Mét00], see similar arguments in [BGRSZ02, Cou11], that the hyperbolic region $\mathcal{H}$ does not contain any glancing point, and $\mathbb{E}^s(\zeta)$ depends analytically on ζ in the neighborhood of any point of $\mathcal{H}$. We now make our weak stability condition precise (recall the notation $B := \mathrm{d}b(0)$).

Assumption 1.6 (Weak Kreiss-Lopatinskii condition)**.**
- *For all $\zeta \in \Xi \setminus \Xi_0$, $\mathrm{Ker}B \cap \mathbb{E}^s(\zeta) = \{0\}$.*
- *The set $\Upsilon := \{\zeta \in \Sigma_0 \,:\, \mathrm{Ker}B \cap \mathbb{E}^s(\zeta) \neq \{0\}\}$ is nonempty and included in the hyperbolic region $\mathcal{H}$.*

- *There exists a neighborhood $\mathcal{V}$ of Υ in Σ, a real valued $\mathcal{C}^\infty$ function σ defined on $\mathcal{V}$, a basis $E_1(\zeta),\dots,E_p(\zeta)$ of $\mathbb{E}^s(\zeta)$ that is of class $\mathcal{C}^\infty$ with respect to $\zeta \in \mathcal{V}$, and a matrix $P(\zeta) \in \mathrm{GL}_p(\mathbb{C})$ that is of class $\mathcal{C}^\infty$ with respect to $\zeta \in \mathcal{V}$, such that*

$$\forall \zeta \in \mathcal{V}\, , \quad B\, \begin{pmatrix} E_1(\zeta) & \dots & E_p(\zeta) \end{pmatrix} = P(\zeta)\,\mathrm{diag}\, \big(\gamma + i\,\sigma(\zeta), 1, \dots, 1\big)\, .$$

As explained in [CG10, CGW14, CW14], Assumption 1.6 is a more convenient description of the so-called WR class of [BGRSZ02]. Let us recall that this class consists of hyperbolic boundary value problems for which the uniform Kreiss-Lopatinskii condition breaks down "at first order" in the hyperbolic region[4]. This class is meaningful for nonlinear problems because it is stable by perturbations of the matrices $A_j(0)$ and of the boundary conditions B.

Our final assumption deals with the phase φ_0 occuring in (1.1).

Assumption 1.7 (Critical phase)**.** *The phase φ_0 in (1.1) is defined by*

$$\varphi_0(t,y) := \underline{\tau}\, t + \underline{\eta} \cdot y\, ,$$

with $(\underline{\tau},\underline{\eta}) \in \Upsilon$. In particular, there holds $(\underline{\tau},\underline{\eta}) \in \mathcal{H}$.

The shock waves problem considered in [MR83] enters the framework defined by Assumptions 1.1, 1.2, 1.3, 1.6 and 1.7 with the additional difficulty that the space domain has a free boundary. The vortex sheets problem considered in [AM87] and the analogous one in [WY14] violate Assumption 1.3 but these problems share all main features which we consider here. We restrict our analysis to fixed noncharacteristic boundaries mostly for convenience and simplicity of notation.

Our main results deal with the existence of *approximate* solutions to (1.1). This is the reason why we only make assumptions on the linearized problem at the origin (1.2), and not on the full nonlinear problem (1.1).

1.3 Main result for wavetrains

In Part I, we consider the nonlinear problem (1.1) with a source term G that is Θ-periodic with respect to its last argument θ_0. As evidenced in several previous works, the asymptotic behavior of the solution u_ε to (1.1) is described in terms of the characteristic phases whose trace on the boundary equals φ_0. We thus consider the pairwise distinct roots (and all the roots are real) $\underline{\omega}_1,\dots,\underline{\omega}_M$ to the dispersion relation

$$\det \Big[\underline{\tau}\, I + \sum_{j=1}^{d-1} \underline{\eta}_j\, A_j(0) + \omega\, A_d(0) \Big] = 0\, .$$

To each $\underline{\omega}_m$ there corresponds a unique integer $k_m \in \{1,\dots,q\}$ such that $\underline{\tau} + \lambda_{k_m}(\underline{\eta},\underline{\omega}_m) = 0$. We can then define the following real phases and their associated group velocity:

$$\forall\, m = 1,\dots,M\, , \quad \varphi_m(t,x) := \varphi(t,y) + \underline{\omega}_m\, x_d\, , \quad \mathbf{v}_m := \nabla \lambda_{k_m}(\underline{\eta},\underline{\omega}_m)\, . \tag{1.3}$$

We let $\Phi := (\varphi_1,\dots,\varphi_M)$ denote the collection of phases. Each group velocity $\mathbf{v}_m$ is either incoming or outgoing with respect to the space domain $\mathbb{R}^d_+$: the last coordinate of $\mathbf{v}_m$ is nonzero. This property holds because $(\underline{\tau},\underline{\eta})$ does not belong to the glancing set $\mathcal{G}$.

[4] Let us also recall that the uniform Kreiss-Lopatinskii condition is satisfied when $\mathrm{Ker} B \cap \mathbb{E}^s(\zeta) = \{0\}$ for all $\zeta \in \Xi$, and not only for $\zeta \in \Xi \setminus \Xi_0$.

Definition 1.8. *The phase φ_m is incoming if the group velocity $\mathbf{v}_m$ is incoming ($\partial_{\xi_d}\lambda_{k_m}(\underline{\eta},\underline{\omega}_m) > 0$), and it is outgoing if the group velocity $\mathbf{v}_m$ is outgoing ($\partial_{\xi_d}\lambda_{k_m}(\underline{\eta},\underline{\omega}_m) < 0$).*

In all what follows, we let $\mathcal{I}$ denote the set of indices $m \in \{1,\dots,M\}$ such that φ_m is incoming, and $\mathcal{O}$ denote the set of indices $m \in \{1,\dots,M\}$ such that φ_m is outgoing. Under Assumption 1.3, both $\mathcal{I}$ and $\mathcal{O}$ are nonempty, as follows from [CG10, Lemma 3.1] which we recall later on.

The proof of our main result for wavetrains, that is Theorem 1.10 below, heavily relies on the nonresonance assumption below. For later use, we introduce the following notation: if $0 \le k \le M$, we let $\mathbb{Z}^{M;k}$ denote the subset of all $\alpha \in \mathbb{Z}^M$ such that at most k coordinates of α are nonzero. For instance $\mathbb{Z}^{M;1}$ is the union of the sets $\mathbb{Z}\,\mathbf{e}_m$, $m = 1,\dots,M$, where $(\mathbf{e}_1,\dots,\mathbf{e}_M)$ denotes the canonical basis of $\mathbb{R}^M$. We also introduce the notation:

$$L(\tau,\xi) := \tau\, I + \sum_{j=1}^d \xi_j\, A_j(0)\,, \quad L(\partial) := \partial_t + \sum_{j=1}^d A_j(0)\,\partial_j\,. \tag{1.4}$$

The nonresonance assumption reads as follows.

Assumption 1.9 (Nonresonance and small divisors condition)**.** *The phases are nonresonant, that is for all $\alpha \in \mathbb{Z}^M \setminus \mathbb{Z}^{M;1}$, there holds* $\det L(\mathrm{d}(\alpha\cdot\Phi)) \neq 0$, *where $\alpha\cdot\Phi := \alpha_m\,\varphi_m$.*

Furthermore, there exists a constant $c > 0$ and a real number γ such that for all $\alpha \in \mathbb{Z}^M \setminus \mathbb{Z}^{M;1}$ that satisfies $\alpha_m = 0$ for all $m \in \mathcal{O}$, there holds

$$|\det L(\mathrm{d}(\alpha\cdot\Phi))| \ge c\,|\alpha|^{-\gamma}\,.$$

Let us note that the small divisors condition is only required for α with nonzero components α_m which correspond to incoming phases. If there is only one incoming phase, then there is no such α with at least two nonzero coordinates, and we do not need any small divisors condition. The reason for this simplification will be explained in Sections 2 and 3. Our main result reads as follows.

Theorem 1.10. *Let Assumptions 1.1, 1.2, 1.3, 1.6, 1.7, 1.9 be satisfied, let $T_0 > 0$ and consider $G \in \mathcal{C}^\infty((-\infty,T_0]_t; H^{+\infty}(\mathbb{R}^{d-1}_y \times (\mathbb{R}/(\Theta\,\mathbb{Z}))_{\theta_0}))$ that vanishes for $t < 0$. Then there exists $0 < T \le T_0$ and there exists a unique sequence of profiles $(\mathcal{U}_n)_{n\ge0}$ in $\mathcal{C}^\infty((-\infty,T]; H^{+\infty}(\mathbb{R}^d_+ \times (\mathbb{R}/(\Theta\,\mathbb{Z}))^M))$ that satisfies the WKB cascade (4.1), (4.3) below, and $\mathcal{U}_n|_{t<0} = 0$ for all $n \in \mathbb{N}$. In particular, each profile $\mathcal{U}_k$ has its θ-spectrum included in the set*

$$\mathbb{Z}^M_{\mathcal{I}} := \big\{\alpha \in \mathbb{Z}^M \,/\, \forall\, m \in \mathcal{O}\,,\ \alpha_m = 0\big\}\,,$$

which means that no outgoing signal is generated at any order.

Furthermore, if for all integers $N_1, N_2 \ge 0$, we define the approximate solution

$$u_\varepsilon^{app,N_1,N_2}(t,x) := \sum_{n=0}^{N_1+N_2} \varepsilon^{1+n}\,\mathcal{U}_n\left(t,x,\frac{\Phi(t,x)}{\varepsilon}\right)\,,$$

then

$$\begin{cases} \partial_t u_\varepsilon^{app,N_1,N_2} + \displaystyle\sum_{j=1}^d A_j(u_\varepsilon^{app,N_1,N_2})\,\partial_j u_\varepsilon^{app,N_1,N_2} = O(\varepsilon^{N_1+1})\,, & t \le T\,,\ x \in \mathbb{R}^d_+\,,\\ b(u_\varepsilon^{app,N_1,N_2}|_{x_d=0}) = \varepsilon^2\, G\left(t,y,\dfrac{\varphi_0(t,y)}{\varepsilon}\right) + O(\varepsilon^{N_1+2})\,, & t \le T\,,\ y \in \mathbb{R}^{d-1}\,,\\ u_\varepsilon^{app,N_1,N_2}|_{t<0} = 0\,, & \end{cases}$$

where the $O(\varepsilon^{N_1+1})$ in the interior equation and $O(\varepsilon^{N_1+2})$ in the boundary conditions are measured respectively in the $\mathcal{C}((-\infty,T];H^{N_2}(\mathbb{R}^d_+))\cap L^\infty((-\infty,T]\times\mathbb{R}^d_+)$ and $\mathcal{C}((-\infty,T];H^{N_2}(\mathbb{R}^{d-1}))\cap L^\infty((-\infty,T]\times\mathbb{R}^{d-1})$ norms.

Of course, the approximate solutions provided by Theorem 1.10 become interesting only for $N_1 \geq 1$, that is, when the remainder $O(\varepsilon^{N_1+2})$ on the boundary becomes smaller than the source term $\varepsilon^2 G$.

The spectrum property in Theorem 1.10 is a rigorous justification of the causality arguments used in [AM87, WY14]. Theorem 1.10 will be proved in Part I of this article. In Section 2, we shall derive the so-called leading amplitude equation from which the leading profile $\mathcal{U}_0$ is constructed. Section 3 is devoted to proving well-posedness for this evolution equation. As far as we know, the bilinear Fourier multiplier that we shall encounter in this equation had not appeared earlier in the geometric optics context and our main task is to prove a tame boundedness estimate for this multiplier. Section 4 is devoted to the construction of the correctors $\mathcal{U}_n$, $n \geq 1$, and to completing the proof of Theorem 1.10. We refer to Appendix A for a discussion of the two-dimensional isentropic Euler equations, with specific emphasis on Assumption 1.9.

1.4 Main result for pulses

We keep the same notation (1.3) for the phases, but now consider the nonlinear problem (1.1) with a source term G that has "polynomial decay" with respect to its last argument θ_0. This behavior is made precise by introducing the following weighted Sobolev spaces.

$$\Gamma^k(\mathbb{R}^d) := \Big\{ u \in L^2(\mathbb{R}^{d-1}_y \times \mathbb{R}_\theta) \, : \, \theta^\alpha \, \partial^\beta_{y,\theta} u \in L^2(\mathbb{R}^d) \quad \text{if } \alpha + |\beta| \leq k \Big\} \, .$$

Theorem 1.11. *Let Assumptions 1.1, 1.2, 1.3, 1.6, 1.7 be satisfied. Let k_0 denote the smallest integer satisfying $k_0 > (d+1)/2$, and let K_0, K_1 be two integers such that $K_0 > 8 + (d+2)/2$, $K_1 - K_0 \geq 2\,k_0 + 2$. Let $T_0 > 0$ and consider*

$$G \in \cap_{\ell=0}^{K_0-1} \mathcal{C}^\ell((-\infty,T_0]_t;\Gamma^{K_1-\ell}(\mathbb{R}^d_{y,\theta}))\, ,$$

that vanishes for $t < 0$. Then there exists $0 < T \leq T_0$ and there exist profiles $\mathcal{U}_0,\mathcal{U}_1,\mathcal{U}_2$ vanishing in $T < 0$ that satisfy the WKB cascade (5.2), (5.4) below. If we define the approximate solution

$$u^{app}_\varepsilon(t,x) := \sum_{n=0}^{2} \varepsilon^{1+n}\, \mathcal{U}_n \left(t,x,\frac{\varphi_0(t,y)}{\varepsilon},\frac{x_d}{\varepsilon} \right) \, ,$$

then

$$\begin{cases} \partial_t u^{app}_\varepsilon + \displaystyle\sum_{j=1}^d A_j(u^{app}_\varepsilon)\, \partial_j u^{app}_\varepsilon = O(\varepsilon^3) \, , & t \leq T \, , \, x \in \mathbb{R}^d_+ \, , \\ b(u^{app}_\varepsilon|_{x_d=0}) = \varepsilon^2 \, G \left(t,y,\dfrac{\varphi_0(t,y)}{\varepsilon} \right) + O(\varepsilon^3) \, , & t \leq T \, , \, y \in \mathbb{R}^{d-1} \, , \\ u^{app}_\varepsilon|_{t<0} = 0 \, , & \end{cases}$$

where the $O(\varepsilon^3)$ in the interior equation and in the boundary conditions are measured respectively in the $\mathcal{C}((-\infty,T];L^2(\mathbb{R}^d_+)) \cap L^\infty((-\infty,T] \times \mathbb{R}^d_+)$ and $\mathcal{C}((-\infty,T];L^2(\mathbb{R}^{d-1})) \cap L^\infty((-\infty,T] \times \mathbb{R}^{d-1})$ norms.

The exact regularity and decay properties of the profiles are given in (5.17).

The approximate solution provided by Theorem 1.11 is constructed, as in [MR83], according to the following line of thought: we expect that the exact solution u_ε to (1.1) admits an asymptotic expansion of the form

$$u_\varepsilon \sim \varepsilon \sum_{k\geq 0} \varepsilon^k \, \mathcal{U}_k \left(t, x, \frac{\varphi_0(t,y)}{\varepsilon}, \frac{x_d}{\varepsilon} \right) ,$$

that is either finite up to an order $K \geq 2$, or infinite. We plug this ansatz and try to identify each profile $\mathcal{U}_k$. The corrector $\varepsilon \, \mathcal{U}_1$ is expected to be negligible with respect to $\mathcal{U}_0$, and so on for higher indices. Hence the identification of the profiles is based on some *boundedness assumption* for the correctors to the leading profile. Of course, such assumptions have to be verified a posteriori when constructing $\mathcal{U}_0, \mathcal{U}_1$ and so on. For instance, Theorem 1.10 is based on the assumption that one can decompose u_ε with profiles in H^∞, and we give a rigorous construction of such profiles for which the corrector

$$\varepsilon^{1+n} \, \mathcal{U}_n \left(t, x, \frac{\Phi(t,x)}{\varepsilon} \right) ,$$

is indeed an $O(\varepsilon^{1+n})$ in L^∞.

In Sections 5 and 6, we give a rigorous construction of the leading profile $\mathcal{U}_0$ and of the first two correctors $\mathcal{U}_1$, $\mathcal{U}_2$ that satisfy all the boundedness and integrability properties on which the derivation of the leading amplitude relies. In particular in section 5 we explain why, assuming that the first and second correctors $\mathcal{U}_1, \mathcal{U}_2$ satisfy some boundedness and integrability properties in the stretched variables (t, x, θ_0, ξ_d), the leading profile $\mathcal{U}_0$ is necessarily determined by an amplitude equation that is entirely similar to the one in [MR83]. The analysis of Section 5 clarifies some of the causality arguments used in [MR83]. This makes the arguments of [MR83] consistent, and one of our achievements is to prove in section 6 local well-posedness for the leading amplitude equation derived in [MR83].

However, the drawback of this approach is that, surprisingly, it is not consistent with the formal large period limit for wavetrains. More precisely, it seems rather reasonable to expect that the pulse problem is obtained by considering the analogous problem for wavetrains with a period Θ and by letting Θ tend to infinity. In particular, the reader can check that the leading amplitude equations derived in [CW13], resp. [CW14], for *quasilinear uniformly stable* pulse problems, resp. *semilinear weakly stable* pulse problems, coincide with the large period limit of the analogous equations obtained in [CGW11], resp. [CGW14], for wavetrains, even though the latter equations include interaction integrals to account for resonances. One could therefore adopt a different point of view and first derive the profile equations for pulses by considering the limit $\Theta \to +\infty$ for wavetrains, and then study the property of the corresponding approximate solution. Surprisingly, the two approaches do not give the same leading profile $\mathcal{U}_0$, as we shall explain in Appendix B. It seems very difficult at this stage to decide which of the two approximate solutions should be the most "physically relevant" since we do not have a nonlinear stability result that would claim that the *exact* solution u_ε to (1.1) is close to one of these two approximate solutions on a fixed time interval independent of ε small enough. The clarification of this surprising phenomenon is left to a future work.

Part I
Highly oscillating wavetrains

2 Construction of approximate solutions: the leading amplitude

2.1 Some decompositions and notation

We recall here some useful results from [CG10] and introduce some notation. Recall that the matrix $\mathcal{A}(\underline{\tau},\underline{\eta})$ is diagonalizable with eigenvalues $i\,\underline{\omega}_m$, $m = 1,\dots,M$. The eigenspace of $\mathcal{A}(\underline{\tau},\underline{\eta})$ for $i\,\underline{\omega}_m$ coincides with the kernel of $L(\mathrm{d}\varphi_m)$.

Lemma 2.1 ([CG10])**.** *The (extended) stable subspace $\mathbb{E}^s(\underline{\tau},\underline{\eta})$ admits the decomposition*

$$\mathbb{E}^s(\underline{\tau},\underline{\eta}) = \oplus_{m\in\mathcal{I}} \,\mathrm{Ker}\; L(\mathrm{d}\varphi_m)\,, \tag{2.1}$$

and each vector space in the decomposition (2.1) is of real type (that is, it admits a basis of real vectors).

Lemma 2.2 ([CG10])**.** *The following decompositions hold*

$$\mathbb{C}^N = \oplus_{m=1}^M \,\mathrm{Ker}\; L(\mathrm{d}\varphi_m) = \oplus_{m=1}^M \,A_d(0)\,\mathrm{Ker}\; L(\mathrm{d}\varphi_m)\,, \tag{2.2}$$

and each vector space in the decompositions (2.2) is of real type.

We let $P_1,\dots,P_M$, resp. $Q_1,\dots,Q_M$, denote the projectors associated with the first, resp. second, decomposition in (2.2). Then for all $m = 1,\dots,M$, there holds $\mathrm{Im}\; L(\mathrm{d}\varphi_m) = \mathrm{Ker}\; Q_m$.

Using Lemma 2.2, we may introduce the partial inverse R_m of $L(\mathrm{d}\varphi_m)$, which is uniquely determined by the relations

$$\forall\, m = 1,\dots,M\,,\quad R_m\,L(\mathrm{d}\varphi_m) = I - P_m\,,\quad L(\mathrm{d}\varphi_m)R_m = I - Q_m,\quad P_m\,R_m = 0\,,\quad R_m\,Q_m = 0\,.$$

When the system is strictly hyperbolic, which is the case considered in Sections 2, 3 and most of Section 4, each vector space $\mathrm{Ker}\; L(\mathrm{d}\varphi_m)$ is one-dimensional and $M = N$. The case of conservative hyperbolic systems with constant multiplicity will be dealt with in Paragraph 4.4. In the case of a strictly hyperbolic system, we choose, for all $m = 1,\dots,N$, a real vector r_m that spans $\mathrm{Ker}\; L(\mathrm{d}\varphi_m)$. We also choose real row vectors $\ell_1,\dots,\ell_N$, that satisfy

$$\forall\, m = 1,\dots,N\,,\quad \ell_m\,L(\mathrm{d}\varphi_m) = 0\,,$$

together with the normalization $\ell_m\,A_d(0)\,r_{m'} = \delta_{mm'}$. With this choice, the partial inverse R_m and the projectors P_m, Q_m are given by

$$\forall\, X \in \mathbb{C}^N\,,\quad R_m\,X = \sum_{m'\neq m} \frac{\ell_{m'}\,X}{\underline{\omega}_m - \underline{\omega}_{m'}}\,r_{m'}\quad P_mX = (\ell_m A_d(0)X)r_m,\quad Q_mX = (\ell_m X)A_d(0)r_m.$$

According to Assumption 1.6, $\mathrm{Ker}B \cap \mathbb{E}^s(\underline{\tau},\underline{\eta})$ is one-dimensional and is therefore spanned by some vector $e = \sum_{m\in\mathcal{I}} e_m$, $e_m \in \mathrm{Span}\; r_m$ (here we have used Lemma 2.1). The vector space $B\,\mathbb{E}^s(\underline{\tau},\underline{\eta})$ is $(p-1)$-dimensional and is of real type. We can therefore write it as the kernel of a real linear form

$$B\,\mathbb{E}^s(\underline{\tau},\underline{\eta}) = \big\{X \in \mathbb{C}^p\,,\ \underline{b}\,X = 0\big\}\,, \tag{2.3}$$

for a suitable nonzero row vector $\underline{b}$. Eventually, we can introduce the partial inverse of the restriction of B to the vector space $\mathbb{E}^s(\underline{\tau},\underline{\eta})$. More precisely, we choose a supplementary vector space of Span e in $\mathbb{E}^s(\underline{\tau},\underline{\eta})$:

$$\mathbb{E}^s(\underline{\tau},\underline{\eta}) = \check{\mathbb{E}}^s(\underline{\tau},\underline{\eta}) \oplus \text{Span } e \,. \tag{2.4}$$

The matrix B then induces an isomorphism from $\check{\mathbb{E}}^s(\underline{\tau},\underline{\eta})$ to the hyperplane $B\,\mathbb{E}^s(\underline{\tau},\underline{\eta})$.

2.2 Strictly hyperbolic systems of three equations

For simplicity of notation, we first explain the derivation of the leading amplitude equation in the case of a 3×3 strictly hyperbolic system. We keep the notation introduced in the previous paragraph, and we make the following assumption.

Assumption 2.3. *The phases φ_1, φ_3 are incoming and φ_2 is outgoing.*

Assumption 2.3 corresponds to the case $p = 2$ in Assumption 1.3 (up to reordering the phases). The only other possibility that is compatible with Assumption 1.3 is $p = 1$, and two phases are outgoing. This case would yield the standard Burgers equation for determining the leading amplitude (see Paragraph 2.4 below for a detailed discussion), so we focus on $p = 2$ which incorporates the main new difficulty.

Let us now derive the WKB cascade for highly oscillating solutions to (1.1). The solution u_ε to (1.1) is assumed to have an asymptotic expansion of the form

$$u_\varepsilon \sim \varepsilon \sum_{k \geq 0} \varepsilon^k \, \mathcal{U}_k \left(t, x, \frac{\Phi(t,x)}{\varepsilon} \right) , \tag{2.5}$$

where we recall that Φ denotes the collection of phases $(\varphi_1, \varphi_2, \varphi_3)$, and the profiles $\mathcal{U}_k$ are assumed to be Θ-periodic with respect to each of their last three arguments $\theta_1, \theta_2, \theta_3$. Plugging the ansatz (2.5) in (1.1) and identifying powers of ε, we obtain the following first three relations for the $\mathcal{U}_k$'s (see Section 4 for the complete set of relations up to any order):

$$\begin{aligned}
&\text{(a)} \quad \mathcal{L}(\partial_\theta)\,\mathcal{U}_0 = 0 \,, \\
&\text{(b)} \quad \mathcal{L}(\partial_\theta)\,\mathcal{U}_1 + L(\partial)\,\mathcal{U}_0 + \mathcal{M}(\mathcal{U}_0,\mathcal{U}_0) = 0 \,, \\
&\text{(c)} \quad \mathcal{L}(\partial_\theta)\,\mathcal{U}_2 + L(\partial)\,\mathcal{U}_1 + \mathcal{M}(\mathcal{U}_0,\mathcal{U}_1) + \mathcal{M}(\mathcal{U}_1,\mathcal{U}_0) + \mathcal{N}_1(\mathcal{U}_0,\mathcal{U}_0) + \mathcal{N}_2(\mathcal{U}_0,\mathcal{U}_0,\mathcal{U}_0) = 0 \,,
\end{aligned} \tag{2.6}$$

where the differential operators $\mathcal{L}, \mathcal{M}, \mathcal{N}_1, \mathcal{N}_2$ are defined by:

$$\begin{aligned}
&\mathcal{L}(\partial_\theta) := L(\mathrm{d}\varphi_m)\,\partial_{\theta_m} \,, \\
&\mathcal{M}(v,w) := \partial_j \varphi_m \, (\mathrm{d}A_j(0) \cdot v)\,\partial_{\theta_m} w \,, \\
&\mathcal{N}_1(v,w) := (\mathrm{d}A_j(0) \cdot v)\,\partial_j w \,, \\
&\mathcal{N}_2(v,v,w) := \frac{1}{2}\,\partial_j \varphi_m \, (\mathrm{d}^2 A_j(0) \cdot (v,v))\,\partial_{\theta_m} w \,.
\end{aligned} \tag{2.7}$$

The equations (2.6) in the domain $(-\infty, T] \times \mathbb{R}^d_+ \times (\mathbb{R}/\Theta\,\mathbb{Z})^3$ are supplemented with the boundary conditions obtained by plugging (2.5) in the boundary conditions of (1.1), which yields (recall $B = \mathrm{d}b(0)$):

$$\begin{aligned}
&\text{(a)} \quad B\,\mathcal{U}_0 = 0 \,, \\
&\text{(b)} \quad B\,\mathcal{U}_1 + \frac{1}{2}\,\mathrm{d}^2 b(0) \cdot (\mathcal{U}_0,\mathcal{U}_0) = G(t,y,\theta_0) \,,
\end{aligned} \tag{2.8}$$

where functions on the left hand side of (2.8) are evaluated at $x_d = 0$ and $\theta_1 = \theta_2 = \theta_3 = \theta_0$. In order to get $u_\varepsilon|_{t<0} = 0$, as required in (1.1), we also look for profiles $\mathcal{U}_k$ that vanish for $t < 0$.

The derivation of the leading amplitude equation is split in several steps, which we decompose below in order to highlight the (slight) differences in Paragraph 4.4 for the case of systems with constant multiplicity.

Step 1: $\mathcal{U}_0$ has mean zero.

According to Assumption 1.9, the phases φ_m are nonresonant. Equation (2.6)(a) thus yields the polarization condition for the leading amplitude $\mathcal{U}_0$. More precisely, we expand the amplitude $\mathcal{U}_0$ in Fourier series with respect to the θ_m's, and (2.6)(a) shows that only the characteristic modes $\mathbb{Z}^{3,1}$ occur in $\mathcal{U}_0$. More precisely, we can write

$$\mathcal{U}_0(t,x,\theta_1,\theta_2,\theta_3) = \underline{\mathcal{U}}_0(t,x) + \sum_{m=1}^{3} \sigma_m(t,x,\theta_m)\, r_m\,, \tag{2.9}$$

with unknown scalar functions σ_m depending on a single periodic variable θ_m and of mean zero with respect to this variable.

Let us now consider Equation (2.6)(b), and integrate with respect to $\theta_1, \theta_2, \theta_3$. Using the expression (2.9) of $\mathcal{U}_0$, we obtain

$$L(\partial)\, \underline{\mathcal{U}}_0 = 0\,, \tag{2.10}$$

because the quadratic term $\mathcal{M}(\mathcal{U}_0, \mathcal{U}_0)$ has zero mean with respect to $(\theta_1, \theta_2, \theta_3)$. Indeed, $\mathcal{M}(\mathcal{U}_0, \mathcal{U}_0)$ splits as the sum of terms that have one of the following three forms :

$$\star)\partial_{\theta_m}\sigma_m\, f_m(t,x)\,, \quad \star)\, \sigma_m\, \partial_{\theta_m}\sigma_m\, \tilde{r}_m\,, \quad \star)\, \sigma_{m_1}\, \partial_{\theta_{m_2}}\sigma_{m_2}\, \tilde{r}_{m_1 m_2}\, (m_1 \neq m_2)\,,$$

where $\tilde{r}_m, \tilde{r}_{m_1 m_2}$ are constant vectors (whose precise expression is useless), and each of these terms has zero mean with respect to $(\theta_1, \theta_2, \theta_3)$. Equation (2.10) is supplemented by the boundary condition obtained by integrating (2.8)(a), that is,

$$B\, \underline{\mathcal{U}}_0|_{x_d=0} = 0\,. \tag{2.11}$$

By the well-posedness result of [Cou05], we get $\underline{\mathcal{U}}_0 \equiv 0$. The goal is now to identify the amplitudes σ_m's in (2.9).

Step 2: $\mathcal{U}_0$ has no outgoing mode.

We first start by showing $\sigma_2 \equiv 0$. We first integrate (2.6)(b) with respect to (θ_1, θ_3) and apply the row vector ℓ_2 (which amounts to applying Q_2), obtaining

$$\ell_2\, L(\partial)(\sigma_2\, r_2) + \ell_2 \left(\frac{1}{\Theta^2} \int_0^\Theta \!\! \int_0^\Theta \mathcal{M}(\mathcal{U}_0, \mathcal{U}_0)\, \mathrm{d}\theta_1\, \mathrm{d}\theta_3 \right) = 0\,.$$

Since there is no resonance among the phases, integration of the quadratic term $\mathcal{M}(\mathcal{U}_0, \mathcal{U}_0)$ with respect to (θ_1, θ_3) only leaves the self-interaction term $\sigma_2\, \partial_{\theta_2}\sigma_2$, and the classical Lax lemma [Lax57] for the linear part[5] $\ell_2\, L(\partial)(\sigma_2\, r_2)$ gives the scalar equation

$$\partial_t \sigma_2 + \mathbf{v}_2 \cdot \nabla_x \sigma_2 + c_2\, \sigma_2\, \partial_{\theta_2}\sigma_2 = 0\,, \quad c_2 := \frac{\partial_j \varphi_2\, \ell_2\, (\mathrm{d}A_j(0) \cdot r_2)\, r_2}{\ell_2\, r_2}\,.$$

[5] In fact, $\ell_2\, L(\partial)(\cdot\, r_2)$ equals $\ell_2\, r_2$ times the transport operator $\partial_t + \mathbf{v}_2 \cdot \nabla_x$, and $\ell_2\, r_2$ does not vanish.

Since $\mathbf{v}_2$ is outgoing and σ_2 vanishes for $t<0$, we obtain $\sigma_2 \equiv 0$.

The above derivation of the interior equation for σ_2 can be performed word for word for the other scalar amplitudes σ_1, σ_3, because $\mathcal{M}(\mathcal{U}_0, \mathcal{U}_0)$ also has zero mean with respect to (θ_1, θ_2) and (θ_2, θ_3). We thus get

$$\partial_t \sigma_m + \mathbf{v}_m \cdot \nabla_x \sigma_m + c_m \, \sigma_m \, \partial_{\theta_m} \sigma_m = 0\,, \quad m = 1, 3\,, \quad c_m := \frac{\partial_j \varphi_m \, \ell_m \, (\mathrm{d}A_j(0) \cdot r_m) \, r_m}{\ell_m \, r_m}\,, \tag{2.12}$$

but we now need to determine the trace of σ_m on the boundary $\{x_d = 0\}$.

Since only σ_1, σ_3 appear in the decomposition (2.9), the leading amplitude $\mathcal{U}_0$ takes values in the stable subspace $\mathbb{E}^s(\underline{\tau}, \underline{\eta})$ (Lemma 2.1), and the boundary condition (2.8)(a) yields

$$\sigma_1(t, y, 0, \theta_0) \, r_1 = a(t, y, \theta_0) \, e_1\,, \quad \sigma_3(t, y, 0, \theta_0) \, r_3 = a(t, y, \theta_0) \, e_3\,,$$

for a single unknown scalar function a of zero mean with respect to its last argument θ_0 (recall that $e = e_1 + e_3$ spans the vector space $\mathbb{E}^s(\underline{\tau}, \underline{\eta}) \cap \mathrm{Ker}\, B$).

Step 3: $\mathcal{U}_1$ has no outgoing mode.

The derivation of the equation that governs the evolution of a comes from analyzing the equations for the first corrector $\mathcal{U}_1$. Since (2.6)(c) is more intricate than the corresponding equation in [CGW14], the analysis is starting here to differ from what we did in our previous work [CGW14]. The first corrector $\mathcal{U}_1$ reads

$$\mathcal{U}_1(t, x, \theta_1, \theta_2, \theta_3) = \underline{\mathcal{U}}_1(t, x) + \sum_{m=1}^{3} \mathcal{U}_1^m(t, x, \theta_m) + \mathcal{U}_1^{\mathrm{nc}}(t, x, \theta_1, \theta_2, \theta_3)\,,$$

where $\underline{\mathcal{U}}_1$ represents the mean value with respect to $(\theta_1, \theta_2, \theta_3)$, each $\mathcal{U}_1^m$ incorporates the θ_m-oscillations and has mean zero, and the spectrum of the "noncharacteristic" part $\mathcal{U}_1^{\mathrm{nc}}$ is a subset of $\mathbb{Z}^3 \setminus \mathbb{Z}^{3,1}$ due to the nonresonant Assumption 1.9. More precisely, $\mathcal{U}_1^{\mathrm{nc}}$ is obtained by expanding (2.6)(b) in Fourier series and retaining only the noncharacteristic modes $\mathbb{Z}^3 \setminus \mathbb{Z}^{3,1}$. From the expression (2.9) of $\mathcal{U}_0$ (recall $\underline{\mathcal{U}}_0 \equiv 0$ and $\sigma_2 \equiv 0$), we get

$$\mathcal{L}(\partial_\theta) \, \mathcal{U}_1^{\mathrm{nc}} = -\sigma_1 \, \partial_{\theta_3} \sigma_3 \, \partial_j \varphi_3 \, (\mathrm{d}A_j(0) \cdot r_1) \, r_3 - \sigma_3 \, \partial_{\theta_1} \sigma_1 \, \partial_j \varphi_1 \, (\mathrm{d}A_j(0) \cdot r_3) \, r_1\,. \tag{2.13}$$

In particular, the spectrum of $\mathcal{U}_1^{\mathrm{nc}}$ is a subset of the integers $\alpha \in \mathbb{Z}^3$ that satisfy $\alpha_2 = 0$ and $\alpha_1 \, \alpha_3 \neq 0$, so $\mathcal{U}_1^{\mathrm{nc}}$ has zero mean when integrated with respect to (θ_1, θ_3).

Equation (2.6)(b) also shows that the component $\mathcal{U}_1^2$ that carries the θ_2-oscillations of $\mathcal{U}_1$ satisfies $L(\mathrm{d}\varphi_2) \, \partial_{\theta_2} \mathcal{U}_1^2 = 0$, so that $\mathcal{U}_1^2$ can be written as $\mathcal{U}_1^2 = \tau_2(t, x, \theta_2) \, r_2$ for an unknown scalar function τ_2 of zero mean with respect to θ_2.

Let us now consider Equation (2.6)(c). Since $\mathcal{U}_0$ only has oscillations in θ_1 and θ_3, and since there is no resonance among the phases, none of the terms $\mathcal{N}_1(\mathcal{U}_0, \mathcal{U}_0)$, $\mathcal{N}_2(\mathcal{U}_0, \mathcal{U}_0, \mathcal{U}_0)$ has oscillations in θ_2 only. Looking also closely at each term in $\mathcal{M}(\mathcal{U}_0, \mathcal{U}_1)$, $\mathcal{M}(\mathcal{U}_1, \mathcal{U}_0)$, we find that both expressions have zero mean with respect to (θ_1, θ_3), because the only way to generate a θ_2-oscillation would be to have a nonzero mode of the form $(\alpha_1, \alpha_2, 0)$ or $(0, \alpha_2, \alpha_3)$ with $\alpha_2 \neq 0$ in $\mathcal{U}_1^{\mathrm{nc}}$, but there is no such mode according to (2.13). We thus derive the outgoing transport equation

$$\partial_t \tau_2 + \mathbf{v}_2 \cdot \nabla_x \tau_2 = 0\,,$$

from which we get $\tau_2 \equiv 0$.

Step 4: computation of the nonpolarized components of $\mathcal{U}_1^1, \mathcal{U}_1^3$, and compatibility condition.

At this stage, we know that the first corrector $\mathcal{U}_1$ reads

$$\mathcal{U}_1 = \mathcal{U}_1(t,x,\theta_1,\theta_3) = \underline{\mathcal{U}}_1(t,x) + \mathcal{U}_1^1(t,x,\theta_1) + \mathcal{U}_1^3(t,x,\theta_3) + \mathcal{U}_1^{\rm nc}(t,x,\theta_1,\theta_3)\,,$$

with $\mathcal{U}_1^{\rm nc}$ determined by (2.13). Moreover, the nonpolarized part of $\mathcal{U}_1^{1,3}$ is obtained by considering Equation (2.6)(b) and retaining only the $\theta_{1,3}$ Fourier modes. We get

$$L(\mathrm{d}\varphi_m)\,\partial_{\theta_m}\mathcal{U}_1^m = -L(\partial)\,(\sigma_m\,r_m) - \sigma_m\,\partial_{\theta_m}\sigma_m\,\partial_j\varphi_m\,(\mathrm{d}A_j(0)\cdot r_m)\,r_m\,,\quad m=1,3\,,$$

so $(I-P_m)\mathcal{U}_1^m$, $m=1,3$, is the only zero mean function that satisfies

$$(I-P_m)\,\partial_{\theta_m}\mathcal{U}_1^m = -R_m\,L(\partial)\,(\sigma_m\,r_m) - \sigma_m\,\partial_{\theta_m}\sigma_m\,\partial_j\varphi_m\,R_m\,(\mathrm{d}A_j(0)\cdot r_m)\,r_m\,,\quad m=1,3\,. \tag{2.14}$$

We now consider the boundary condition (2.8)(b), which we rewrite equivalently as:

$$\begin{aligned}B\,\underline{\mathcal{U}}_1|_{x_d=0} + B\,P_1\,\mathcal{U}_1^1|_{x_d=0,\theta_1=\theta_0} + B\,P_3\,\mathcal{U}_1^3|_{x_d=0,\theta_3=\theta_0}&\\ +B\,(I-P_1)\,\mathcal{U}_1^1|_{x_d=0,\theta_1=\theta_0} + B\,(I-P_3)\,\mathcal{U}_1^3|_{x_d=0,\theta_3=\theta_0} + B\,\mathcal{U}_1^{\rm nc}|_{x_d=0,\theta_1=\theta_3=\theta_0}&\\ +\frac{1}{2}\,(\mathrm{d}^2b(0)\cdot(e,e))\,a^2 = G(t,y,\theta_0)\,.&\end{aligned}$$

We differentiate the latter equation with respect to θ_0 and apply the row vector $\underline{b}$, so that the first line vanishes. We are left with

$$\begin{aligned}\underline{b}\,B\,(I-P_1)\,(\partial_{\theta_1}\mathcal{U}_1^1)|_{x_d=0,\theta_1=\theta_0} + \underline{b}\,B\,(I-P_3)\,(\partial_{\theta_3}\mathcal{U}_1^3)|_{x_d=0,\theta_3=\theta_0} + \underline{b}\,B\,\partial_{\theta_0}(\mathcal{U}_1^{\rm nc}|_{x_d=0,\theta_1=\theta_3=\theta_0})&\\ +\frac{1}{2}\,\underline{b}\,(\mathrm{d}^2b(0)\cdot(e,e))\,\partial_{\theta_0}(a^2) = \underline{b}\,\partial_{\theta_0}G\,.&\end{aligned}$$

The first two terms in the first row are computed by using (2.14), and [CG10, Proposition 3.5]. We get

$$\upsilon\,\partial_{\theta_0}(a^2) - X_{\rm Lop}a + \underline{b}\,B\,\partial_{\theta_0}(\mathcal{U}_1^{\rm nc}|_{x_d=0,\theta_1=\theta_3=\theta_0}) = \underline{b}\,\partial_{\theta_0}G\,, \tag{2.15}$$

where the constant υ and the vector field $X_{\rm Lop}$ are defined by

$$\upsilon := \frac{1}{2}\,\underline{b}\,(\mathrm{d}^2b(0)\cdot(e,e)) - \frac{1}{2}\,\underline{b}\,B\,R_1\,\partial_j\varphi_1\,(\mathrm{d}A_j(0)\cdot e_1)\,e_1 - \frac{1}{2}\,\underline{b}\,B\,R_3\,\partial_j\varphi_3\,(\mathrm{d}A_j(0)\cdot e_3)\,e_3\,, \tag{2.16}$$

$$X_{\rm Lop} := \underline{b}\,B\,(R_1\,e_1 + R_3\,e_3)\,\partial_t + \sum_{j=1}^{d-1}\underline{b}\,B\,(R_1\,A_j(0)\,e_1 + R_3\,A_j(0)\,e_3)\,\partial_j = \iota\,(\partial_\tau\sigma(\underline{\tau},\underline{\eta})\,\partial_t + \partial_{\eta_j}\sigma(\underline{\tau},\underline{\eta})\,\partial_j)\,, \tag{2.17}$$

with ι a nonzero real constant, and σ defined in Assumption 1.6. It is also shown in [CG10, Proposition 3.5] that the partial derivative $\partial_\tau\sigma(\underline{\tau},\underline{\eta})$ does not vanish, so that, up to a nonzero constant, $X_{\rm Lop} = \partial_t + \mathbf{w}\cdot\nabla_y$ for some vector $\mathbf{w}\in\mathbb{R}^{d-1}$ (which represents the group velocity associated with the characteristic set of the Lopatinskii determinant).

Step 5: computation of $\mathcal{U}_1^{\rm nc}$ and conclusion.

The final step in the analysis is to compute the derivative $\partial_{\theta_0}(\mathcal{U}_1^{\rm nc}|_{x_d=0,\theta_1=\theta_3=\theta_0})$ arising in (2.15) in terms of the amplitude a. Restricting (2.13) to the boundary $\{x_d=0\}$ gives

$$\begin{aligned}\mathcal{L}(\partial_\theta)\,\mathcal{U}_1^{\rm nc}|_{x_d=0} = &-a(t,y,\theta_1)\,(\partial_{\theta_0}a)(t,y,\theta_3)\,\partial_j\varphi_3\,({\rm d}A_j(0)\cdot e_1)\,e_3\\ &-a(t,y,\theta_3)\,(\partial_{\theta_0}a)(t,y,\theta_1)\,\partial_j\varphi_1\,({\rm d}A_j(0)\cdot e_3)\,e_1\,.\end{aligned}$$

Let us expand a in Fourier series with respect to θ_0 (recall that a has mean zero):

$$a(t,y,\theta_0)=\sum_{k\in\mathbb{Z}^*}a_k(t,y)\,{\rm e}^{2\,i\,\pi\,k\,\theta_0/\Theta}\,.$$

Then the Fourier series of $\mathcal{U}_1^{\rm nc}$ reads

$$\mathcal{U}_1^{\rm nc}(t,x,\theta_1,\theta_3)=\sum_{k_1,k_3\in\mathbb{Z}^*}u_{k_1,k_3}(t,x)\,{\rm e}^{2\,i\,\pi\,(k_1\,\theta_1+k_3\,\theta_3)/\Theta}\,,$$

with

$$\begin{aligned}&u_{k_1,k_3}(t,y,0)=-L(k_1\,{\rm d}\varphi_1+k_3\,{\rm d}\varphi_3)^{-1}\,(k_1\,E_{1,3}+k_3\,E_{3,1})\,,\\ &E_{1,3}:=\partial_j\varphi_1\,({\rm d}A_j(0)\cdot e_3)\,e_1\,,\quad E_{3,1}:=\partial_j\varphi_3\,({\rm d}A_j(0)\cdot e_1)\,e_3\,.\end{aligned} \tag{2.18}$$

Plugging the latter expression in (2.15), we end up with the evolution equation that governs the leading amplitude a on the boundary:

$$\upsilon\,\partial_{\theta_0}(a^2)-X_{\rm Lop}a+\partial_{\theta_0}Q_{\rm per}[a,a]=\underline{b}\,\partial_{\theta_0}G\,, \tag{2.19}$$

with

$$Q_{\rm per}[a,\widetilde{a}]:=-\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{k_1+k_3=k,\\ k_1\,k_3\neq0}}\underline{b}\,B\,L(k_1\,{\rm d}\varphi_1+k_3\,{\rm d}\varphi_3)^{-1}\,(k_1\,E_{1,3}+k_3\,E_{3,1})\,a_{k_1}\,\widetilde{a}_{k_3}\right)\,{\rm e}^{2\,i\,\pi\,k\,\theta_0/\Theta}\,.$$

Equation (2.19) is a closed equation for the leading amplitude a on the boundary. It involves the vector field $X_{\rm Lop}$ associated with a characteristic of the Lopatinskii determinant, a Burgers term $\partial_{\theta_0}a^2$ and a new quadratic nonlinearity $\partial_{\theta_0}Q_{\rm per}[a,a]$. The operator $Q_{\rm per}$ takes the form of a bilinear Fourier multiplier. Its above expression may be simplified a little bit by computing the decomposition of $L(k_1\,{\rm d}\varphi_1+k_3\,{\rm d}\varphi_3)^{-1}$ on the basis r_1,r_2,r_3, and by recalling the property $\underline{b}\,B\,r_1=\underline{b}\,B\,r_3=0$ (so only the component of $L(k_1\,{\rm d}\varphi_1+k_3\,{\rm d}\varphi_3)^{-1}$ on r_2 matters). We obtain:

$$Q_{\rm per}[a,\widetilde{a}]:=-\underline{b}\,B\,r_2\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{k_1+k_3=k,\\ k_1\,k_3\neq0}}\frac{k_1\,\ell_2\,E_{1,3}+k_3\,\ell_2\,E_{3,1}}{k_1\,(\underline{\omega}_1-\underline{\omega}_2)+k_3\,(\underline{\omega}_3-\underline{\omega}_2)}\,a_{k_1}\,\widetilde{a}_{k_3}\right)\,{\rm e}^{2\,i\,\pi\,k\,\theta_0/\Theta}\,. \tag{2.20}$$

Anticipating slightly our discussion in Section 3, well-posedness of (2.19) will be linked to arithmetic properties of the phases φ_m, and this is one reason for the small divisors condition in Assumption 1.9. This is in sharp contrast with the theory of *weakly nonlinear* geometric optics for both the Cauchy problem (see [HMR86, JMR93, JMR95] and references therein) and for uniformly stable boundary value problems (see [Wil99, Wil02, CGW11]), where arithmetic properties of the phases do not enter the discussion on the leading profile for the high frequency limit.

2.3 Extension to strictly hyperbolic systems of size N

The above derivation of Equation (2.19) can be extended without any difficulty to the case of a hyperbolic system of size N provided that Assumption 1.9 is satisfied. In that case, the number of phases equals N.

Steps 1 and 2 of the above analysis extends almost word for word, to the price of changing some notation. Namely, the first relations of the WKB cascade (2.6), (2.8) shows that the leading amplitude $\mathcal{U}_0$ reads

$$\mathcal{U}_0(t,x,\theta_1,\dots,\theta_N) = \underline{\mathcal{U}}_0(t,x) + \sum_{m=1}^{N} \sigma_m(t,x,\theta_m)\, r_m \,,$$

with unknown scalar functions σ_m depending on a single periodic variable θ_m and of mean zero with respect to this variable. The quadratic expression $\mathcal{M}(\mathcal{U}_0,\mathcal{U}_0)$ still has zero mean with respect to $(\theta_1,\dots,\theta_N)$ so the nonoscillating part $\underline{\mathcal{U}}_0$ satisfies (2.10) and (2.11), and therefore vanishes. Furthermore, each function σ_m satisfies the Burgers equation (2.12), which reduces the leading amplitude $\mathcal{U}_0$ to

$$\mathcal{U}_0(t,x,\theta_1,\dots,\theta_N) = \sum_{m\in\mathcal{I}} \sigma_m(t,x,\theta_m)\, r_m \,, \tag{2.21}$$

where $\mathcal{I}$ denotes the set of incoming phases. The boundary condition (2.8)(a) then yields

$$\forall\, m \in \mathcal{I}\,,\quad \sigma_m(t,y,0,\theta_0)\, r_m = a(t,y,\theta_0)\, e_m \,,$$

for a single unknown scalar function a of zero mean with respect to its last argument θ_0 ($e = \sum_{m\in\mathcal{I}} e_m$ spans the vector space $\mathbb{E}^s(\underline{\tau},\underline{\eta}) \cap \operatorname{Ker} B$).

Step 3 of the above discussion is unchanged, showing that in the first corrector $\mathcal{U}_1$, each profile $\mathcal{U}_1^m$ vanishes when the index m corresponds to an outgoing phase. The noncharacteristic part $\mathcal{U}_1^{\rm nc}$ is obtained by using the relation

$$\mathcal{L}(\partial_\theta)\,\mathcal{U}_1^{\rm nc} = -\sum_{\substack{m_1<m_2\\ m_1,m_2\in\mathcal{I}}} \sigma_{m_1}\,\partial_{\theta_{m_2}}\sigma_{m_2}\,\partial_j\varphi_{m_2}\,({\rm d}A_j(0)\cdot r_{m_1})\, r_{m_2} + \sigma_{m_2}\,\partial_{\theta_{m_1}}\sigma_{m_1}\,\partial_j\varphi_{m_1}\,({\rm d}A_j(0)\cdot r_{m_2})\, r_{m_1} \,.$$

which is the analogue of (2.13).

Step 4 is also unchanged because $\mathcal{U}_1$ has no outgoing mode, and when m corresponds to an incoming phase, $(I-P_m)\,\partial_{\theta_m}\mathcal{U}_1 m$ is given by (2.14). Eventually, the boundary condition (2.8)(b) gives the compatibility condition

$$\upsilon\,\partial_{\theta_0}(a^2) - X_{\rm Lop} a + \partial_{\theta_0} Q_{\rm per}[a,a] = \underline{b}\,\partial_{\theta_0} G \,, \tag{2.22}$$

with

$$\upsilon := \frac{1}{2}\,\underline{b}\,({\rm d}^2 b(0)\cdot(e,e)) - \frac{1}{2}\sum_{m\in\mathcal{I}} \underline{b}\, B\, R_m\,\partial_j\varphi_m\,({\rm d}A_j(0)\cdot e_m)\, e_m \,, \tag{2.23}$$

$$X_{\rm Lop} := \sum_{m\in\mathcal{I}} \underline{b}\, B\, R_m\, e_m\,\partial_t + \sum_{j=1}^{d-1}\sum_{m\in\mathcal{I}} \underline{b}\, B\, R_m\, A_j(0)\, e_m\,\partial_j = \iota\,(\partial_\tau\sigma(\underline{\tau},\underline{\eta})\,\partial_t + \partial_{\eta_j}\sigma(\underline{\tau},\underline{\eta})\,\partial_j) \,,$$

where ι is a nonzero real constant and the function σ is defined in Assumption 1.6. (Again, [CG10, Proposition 3.5] shows that the partial derivative $\partial_\tau\sigma(\underline{\tau},\underline{\eta})$ does not vanish.) The new expression of the bilinear Fourier multiplier $Q_{\rm per}$ reads:

$$
(2.24)\quad Q_{\rm per}[a,\widetilde{a}] := -\sum_{m\in\mathcal{O}} \underline{b}\,B\,r_m \sum_{\substack{m_1<m_2\\ m_1,m_2\in\mathcal{I}}} \sum_{k\in\mathbb{Z}} \left(\sum_{\substack{k_{m_1}+k_{m_2}=k,\\ k_{m_1}\,k_{m_2}\neq 0}} \frac{k_{m_1}\,\ell_m\,E_{m_1,m_2}+k_{m_2}\,\ell_m\,E_{m_2,m_1}}{k_{m_1}\,(\underline{\omega}_{m_1}-\underline{\omega}_m)+k_{m_2}\,(\underline{\omega}_{m_2}-\underline{\omega}_m)}\,a_{k_{m_1}}\,\widetilde{a}_{k_{m_2}} \right) {\rm e}^{2\,i\,\pi\,k\,\theta_0/\Theta}\,,
$$

with

$$
(2.25)\qquad \forall\, m_1,m_2\in\mathcal{I}\,,\quad E_{m_1,m_2} := \partial_j\varphi_{m_1}\,({\rm d}A_j(0)\cdot e_{m_2})\,e_{m_1}\,.
$$

The definition (2.24) reduces to (2.20) when $N=3$ and Assumption 2.3 is satisfied.

2.4 The case with a single incoming phase

In this short paragraph, we explain why the computations in [AM87, WY14] lead to the standard Burgers equation for determining the leading amplitude, and does not incorporate any quadratic nonlinearity of the form (2.20) we have found under Assumption 2.3.

The vortex sheets problem considered in [AM87] and the analogous one in [WY14] differ from the framework that we consider here by the fact that the (free) boundary is characteristic. Nevertheless, one can reproduce a similar normal modes analysis for trying to detect violent or neutral instabilities. The *two-dimensional supersonic* regime considered in [AM87] precludes violent instabilities, but a similar situation to the one encoded in Assumption 1.6 occurs[6]. The situation in [WY14] for three dimensional steady flows is similar, and the corresponding Lopatinskii determinant is computed in [WY13].

The two-dimensional Euler equations form a system of three equations ($N=3$), but due to the characteristic boundary (the corresponding matrix $A_d(0)$ has a kernel of dimension 1), the number of phases φ_m on either side of the vortex sheet equals 2. One of them is incoming, and the other is outgoing. In such a situation, there are too few incoming phases to create a nontrivial component $\mathcal{U}_1^{\rm nc}$ for the first corrector $\mathcal{U}_1$, so that the bilinear Fourier multiplier $Q_{\rm per}$ vanishes. Though our argument is somehow formal, the reader can follow the computations in [AM87] or in [WY14] and check that they follow the exact same procedure that we have described in our general framework.

3 Analysis of the leading amplitude equation

Our goal in this section is to prove a well-posedness result for the leading amplitude equation (2.19). Up to dividing by nonzero constants, and using the shorter notation θ instead of θ_0, the equation takes the form

$$
(3.1)\qquad \partial_t a+\mathbf{w}\cdot\nabla_y a+c\,a\,\partial_\theta a+\mu\,\partial_\theta Q_{\rm per}[a,a]=g\,,
$$

[6]The reader will find in [CS04] a detailed analysis of the roots of the associated Lopatinskii determinant, showing that they are located in the hyperbolic region and simple.

where $\mathbf{w}$ is a fixed vector in $\mathbb{R}^{d-1}$, c, μ are real constants, and $Q_{\rm per}$ is a bilinear Fourier multiplier with respect to the periodic variable θ:

$$Q_{\rm per}[a,a] := \sum_{k \in \mathbb{Z}} \left(\sum_{\substack{k_1+k_3=k, \\ k_1\, k_3 \neq 0}} \frac{k_1\, \ell_2\, E_{1,3} + k_3\, \ell_2\, E_{3,1}}{k_1\, (\underline{\omega}_1 - \underline{\omega}_2) + k_3\, (\underline{\omega}_3 - \underline{\omega}_2)}\, a_{k_1}\, a_{k_3} \right) {\rm e}^{2\, i\, \pi\, k\, \theta/\Theta} \, . \tag{3.2}$$

The source term g in (3.1) belongs to $H^{+\infty}((-\infty, T_0]_t \times \mathbb{R}^{d-1}_y \times (\mathbb{R}/(\Theta\, \mathbb{Z}))_\theta)$, $T_0 > 0$, and vanishes for $t < 0$. Furthermore, it has mean zero with respect to the variable θ. Recall that in (3.2), a_k denotes the k-th Fourier coefficient of a with respect to θ (which is a function of (t, y)).

Recall that for strictly hyperbolic systems of size N, (2.24) should be substituted for (3.2) in the definition of $Q_{\rm per}$, while in the particular case $p = 1$ (one single incoming phase), (3.1) reduces to the standard Burgers equation for which our main well-posedness result, Theorem 3.4 below, is well-known. For simplicity, we thus encompass all cases by studying (3.1), (3.2) and leave to the reader the very minor modifications required for the general case.

3.1 Preliminary reductions

We first introduce the nonzero parameters:

$$\delta_1 := \frac{\underline{\omega}_1 - \underline{\omega}_2}{\underline{\omega}_3 - \underline{\omega}_1} \, , \quad \delta_3 := \frac{\underline{\omega}_3 - \underline{\omega}_2}{\underline{\omega}_3 - \underline{\omega}_1} \, ,$$

that satisfy $\delta_3 = 1 + \delta_1$, and we observe that $Q_{\rm per}[a,a]$ in (3.2) can be written as

$$Q_{\rm per}[a,a] = -\frac{\Theta\, \ell_2\, E_{1,3}}{2\, \pi\, (\underline{\omega}_3 - \underline{\omega}_1)}\, \mathbb{F}_{\rm per}(\partial_\theta a, a) - \frac{\Theta\, \ell_2\, E_{3,1}}{2\, \pi\, (\underline{\omega}_3 - \underline{\omega}_1)}\, \mathbb{F}_{\rm per}(a, \partial_\theta a) \, ,$$

where the bilinear operator $\mathbb{F}_{\rm per}$ is defined by:

$$\mathbb{F}_{\rm per}(u,v) := \sum_{k \in \mathbb{Z}} \left(\sum_{\substack{k_1+k_3=k, \\ k_1\, k_3 \neq 0}} \frac{i\, u_{k_1}\, v_{k_3}}{k_1\, \delta_1 + k_3\, \delta_3} \right) {\rm e}^{2\, i\, \pi\, k\, \theta/\Theta} \, . \tag{3.3}$$

The bilinear operator $\mathbb{F}_{\rm per}$ satisfies the following two properties:

$$\text{(Differentiation)} \quad \partial_\theta(\mathbb{F}_{\rm per}(u,v)) = \mathbb{F}_{\rm per}(\partial_\theta u, v) + \mathbb{F}_{\rm per}(u, \partial_\theta v) \, , \tag{3.4}$$

$$\text{(Integration by parts)} \quad \mathbb{F}_{\rm per}(u, \partial_\theta v) = -\frac{2\, \pi}{\Theta\, \delta_3}\, u\, v - \frac{\delta_1}{\delta_3}\, \mathbb{F}_{\rm per}(\partial_\theta u, v) \, , \quad \text{if } u_0 = v_0 = 0 \, . \tag{3.5}$$

Using the properties (3.4), (3.5), we can rewrite equation (3.1) as

$$\partial_t a + \mathbf{w} \cdot \nabla_y a + c\, a\, \partial_\theta a + \mu\, \mathbb{F}_{\rm per}(\partial_\theta a, \partial_\theta a) = g \, , \tag{3.6}$$

with new (harmless) constants c, μ for which we keep the same notation. Our goal is to solve equation (3.6), that is equivalent to (3.1), by a standard fixed point argument. The main ingredient in the proof is to show that the nonlinear term $\mathbb{F}_{\rm per}(\partial_\theta a, \partial_\theta a)$ acts as a *semilinear* term in the scale of Sobolev spaces.

3.2 Tame boundedness of the bilinear operator $\mathbb{F}_{\rm per}$

The operator $\mathbb{F}_{\rm per}$ is not symmetric but changing the roles of δ_1 and δ_3, the roles of the first and second argument of $\mathbb{F}_{\rm per}$ in the estimates below can be exchanged. This will be used at one point in the analysis below. We let $H^\nu := H^\nu(\mathbb{R}^{d-1} \times (\mathbb{R}/\Theta\,\mathbb{Z}))$ denote the standard Sobolev space of index $\nu \in \mathbb{N}$. The norm is denoted $\| \cdot \|_{H^\nu}$. Functions are assumed to take real values. In the proof of Theorem 3.1 below, we shall also make use of fractional Sobolev spaces on the torus $\mathbb{R}/\Theta\,\mathbb{Z}$ or on the whole space $\mathbb{R}^{d-1}$. These are defined by means of the Fourier transform, see, e.g., [BGS07, BCD11]. Our main boundedness result for the operator $\mathbb{F}_{\rm per}$ reads as follows.

Theorem 3.1. *There exists an integer ν_0 such that, for all $\nu \ge \nu_0$, there exists a constant C_ν satisfying*

$$\forall\, u, v \in H^\nu \,, \quad \|\mathbb{F}_{\rm per}(\partial_\theta u, \partial_\theta v)\|_{H^\nu} \le C_\nu \left(\|u\|_{H^{\nu_0}} \, \|v\|_{H^\nu} + \|u\|_{H^\nu} \, \|v\|_{H^{\nu_0}} \right) . \tag{3.7}$$

Estimate (3.7) is tame because the integer ν_0 is fixed and the right hand side of the inequality depends linearly on the norms $\|u\|_{H^\nu}, \|v\|_{H^\nu}$. This will be used in the proof of Theorem 3.4 below for propagating the regularity of the initial condition for (3.6) on a fixed time interval.

Proof. We first observe that, provided that $\mathbb{F}_{\rm per}(u,v)$ makes sense, then $\mathbb{F}_{\rm per}(u,v)$ takes real values. This simply follows from observing that

$$(\mathbb{F}_{\rm per}(u,v))_{-k} = \overline{(\mathbb{F}_{\rm per}(u,v))_k} \,,$$

provided that u and v take real values. We are now going to prove a convenient new formulation of Assumption 1.9.

Lemma 3.2. *There exist a constant $C > 0$ and a real number $\gamma_0 \ge 0$ such that*

$$\forall\, k_1, k_3 \in \mathbb{Z} \setminus \{0\} \,, \quad \frac{1}{|k_1\, \delta_1 + k_3\, \delta_3|} \le C\, \min(|k_1|^{\gamma_0}, |k_3|^{\gamma_0}) \,.$$

Proof of Lemma 3.2. Since in our framework, we have $\mathcal{I} = \{1,3\}$ and $\mathcal{O} = \{2\}$, we can apply Assumption 1.9 to any $(k_1, 0, k_3) \in \mathbb{Z}^3$ with $k_1\, k_3 \neq 0$. We compute

$$L({\rm d}(k_1\, \varphi_1 + k_3\, \varphi_3))\, r_2 = (k_1\, (\underline{\omega}_1 - \underline{\omega}_2) + k_3\, (\underline{\omega}_3 - \underline{\omega}_2))\, A_d(0)\, r_2 \,,$$

and the quantity $k_1\, (\underline{\omega}_1 - \underline{\omega}_2) + k_3\, (\underline{\omega}_3 - \underline{\omega}_2)$ cannot vanish for otherwise there would be a nonzero vector in the kernel of $L({\rm d}(k_1\, \varphi_1 + k_3\, \varphi_3))$. We thus derive the bound

$$\frac{1}{|k_1\, (\underline{\omega}_1 - \underline{\omega}_2) + k_3\, (\underline{\omega}_3 - \underline{\omega}_2)|} \le C\, \|L({\rm d}(k_1\, \varphi_1 + k_3\, \varphi_3))^{-1}\| \,,$$

for a suitable constant C that does not depend on k_1, k_3. The norm of $L({\rm d}(k_1\, \varphi_1 + k_3\, \varphi_3))^{-1}$ is estimated by combining the lower bound given in Assumption 1.9 for the determinant, and an obvious polynomial bound for the transpose of the comatrix. We have thus shown that there exists a constant $C > 0$ and a real parameter γ_0 (which can be chosen nonnegative without loss of generality), that do not depend on k_1, k_3, such that

$$\frac{1}{|k_1\, (\underline{\omega}_1 - \underline{\omega}_2) + k_3\, (\underline{\omega}_3 - \underline{\omega}_2)|} \le C\, |(k_1, k_3)|^{\gamma_0} \,.$$

Up to changing the constant C, we can rephrase this estimate in terms of the rescaled parameters δ_1, δ_3:

$$\frac{1}{|k_1\,\delta_1 + k_3\,\delta_3|} \leq C\,|(k_1,k_3)|^{\gamma_0}\,, \tag{3.8}$$

and it only remains to substitute the minimum of $|k_1|, |k_3|$ for the norm $|(k_1,k_3)|$ in the right hand side of (3.8).

There are two cases. Either $|k_1\,\delta_1 + k_3\,\delta_3| > |\delta_1| > 0$, and in that case, it is sufficient to choose $C \geq 1/|\delta_1|$ (we use $\gamma_0 \geq 0$). Or $|k_1\,\delta_1 + k_3\,\delta_3| \leq |\delta_1|$, and we have

$$|k_3| \leq \frac{1}{|\delta_3|}\,|k_1\,\delta_1 + k_3\,\delta_3| + \frac{1}{|\delta_3|}\,|k_1\,\delta_1| \leq 2\,\frac{|\delta_1|}{|\delta_3|}\,|k_1|\,,$$

because k_1 is nonzero. Up to choosing a new constant C, (3.8) reduces to

$$\frac{1}{|k_1\,\delta_1 + k_3\,\delta_3|} \leq C\,|k_1|^{\gamma_0}\,,$$

and we can prove the analogous estimate with k_3 instead of k_1 by the same arguments. This completes the proof of Lemma 3.2 $\square$

The proof of Theorem 3.1 relies on the following straightforward extension of [RR82, Lemma 1.2.2]. The proof of Lemma 3.3 is exactly the same as that of [RR82, Lemma 1.2.2], and is therefore omitted.

Lemma 3.3. *Let $\mathbb{K} : \mathbb{R}^{d-1} \times \mathbb{Z} \times \mathbb{R}^{d-1} \times \mathbb{Z} \to \mathbb{C}$ be a locally integrable measurable function such that, either*

$$\sup_{(\xi,k)\in\mathbb{R}^{d-1}\times\mathbb{Z}} \int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} |\mathbb{K}(\xi,k,\eta,\ell)|^2\,\mathrm{d}\eta < +\infty\,,$$

or

$$\sup_{(\eta,\ell)\in\mathbb{R}^{d-1}\times\mathbb{Z}} \int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} |\mathbb{K}(\xi,k,\eta,\ell)|^2\,\mathrm{d}\xi < +\infty\,.$$

Then the map

$$(f,g) \longmapsto \int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} \mathbb{K}(\xi,k,\eta,\ell)\,f(\xi-\eta,k-\ell)\,g(\eta,\ell)\,\mathrm{d}\eta\,,$$

is bounded on $L^2(\mathbb{R}^{d-1}\times\mathbb{Z}) \times L^2(\mathbb{R}^{d-1}\times\mathbb{Z})$ with values in $L^2(\mathbb{R}^{d-1}\times\mathbb{Z})$.

To prove boundedness of the bilinear operator $\mathbb{F}_{\mathrm{per}}(\partial_\theta\cdot,\partial_\theta\cdot)$, we shall apply Lemma 3.3 in the Fourier variables. More precisely, for functions u, v that have finitely Fourier modes in the variable θ and belong to the Schwartz space $\mathcal{S}(\mathbb{R}^{d-1}\times(\mathbb{R}/\Theta\,\mathbb{Z}))$, there holds[7]:

$$c_k(\widehat{\mathbb{F}_{\mathrm{per}}(\partial_\theta u, \partial_\theta v)})(\xi) = \mathrm{C}^{\mathrm{st}} \int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z},\ell\notin\{0,k\}} \frac{(k-\ell)\,\ell}{(k-\ell)\,\delta_1 + \ell\,\delta_3}\,\widehat{c_{k-\ell}(u)}(\xi-\eta)\,\widehat{c_\ell(v)}(\eta)\,\mathrm{d}\eta\,.$$

Omitting from now on the constant multiplicative factor, we consider the symbol

$$\mathbf{K}(k,\ell) := \begin{cases} (k-\ell)\,\ell/((k-\ell)\,\delta_1 + \ell\,\delta_3) & \text{if } \ell \notin \{0,k\}\,, \\ 0 & \text{otherwise.} \end{cases}$$

[7]Here we use the notation c_k for the k-th Fourier coefficient with respect to the variable θ, and the "hat" notation for the Fourier transform with respect to the variable y.

We wish to bound the H^ν norm:

$$\int_{\mathbb{R}^{d-1}} \sum_{k\in\mathbb{Z}} \langle(\xi,k)\rangle^{2\,\nu} \, |c_k(\widehat{\mathbb{F}_{\rm per}(\partial_\theta u, \partial_\theta v)})(\xi)|^2 \, {\rm d}\xi \,,$$

for $\nu \in \mathbb{N}$ large enough ($\langle\cdot\rangle$ stands as usual for the Japanese bracket).

Given the parameter $\gamma_0 \geq 0$ in Lemma 3.2, we fix an integer $\nu_0 > \gamma_0 + 2 + d/2$. We consider two functions χ_1, χ_2 on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\chi_1 + \chi_2 \equiv 1$, and

$$\chi_1(\xi,k,\eta,\ell) = 0 \quad \text{if } \langle(\eta,\ell)\rangle \geq (2/3)\, \langle(\xi,k)\rangle \,,$$
$$\chi_2(\xi,k,\eta,\ell) = 0 \quad \text{if } \langle(\eta,\ell)\rangle \leq (1/3)\, \langle(\xi,k)\rangle \,.$$

We first consider the quantity

$$\int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} \chi_1(\xi,k,\eta,\ell) \, \langle(\xi,k)\rangle^\nu \, \mathbf{K}(k,\ell) \, \widehat{c_{k-\ell}(u)}(\xi-\eta) \, \widehat{c_\ell(v)}(\eta) \, {\rm d}\eta \,, \tag{3.9}$$

which we rewrite as

$$\int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} \frac{\chi_1(\xi,k,\eta,\ell) \, \langle(\xi,k)\rangle^\nu \, \mathbf{K}(k,\ell)}{\langle(\xi-\eta,k-\ell)\rangle^\nu \, \langle(\eta,\ell)\rangle^{\nu_0}} \Big(\langle(\xi-\eta,k-\ell)\rangle^\nu \, \widehat{c_{k-\ell}(u)}(\xi-\eta) \Big) \Big(\langle(\eta,\ell)\rangle^{\nu_0} \, \widehat{c_\ell(v)}(\eta) \Big) \, {\rm d}\eta \,.$$

On the support of χ_1, there holds

$$\langle(\xi-\eta,k-\ell)\rangle \geq \langle(\xi,k)\rangle - \langle(\eta,\ell)\rangle \geq \frac{1}{3} \, \langle(\xi,k)\rangle \,,$$

and therefore

$$\int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} \left| \frac{\chi_1(\xi,k,\eta,\ell) \, \langle(\xi,k)\rangle^\nu \, \mathbf{K}(k,\ell)}{\langle(\xi-\eta,k-\ell)\rangle^\nu \, \langle(\eta,\ell)\rangle^{\nu_0}} \right|^2 \, {\rm d}\eta \leq C \int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} \frac{|\mathbf{K}(k,\ell)|^2}{\langle(\eta,\ell)\rangle^{2\,\nu_0}} \, {\rm d}\eta \,.$$

We now use Lemma 3.2 to derive the bound

$$\left| \frac{(k-\ell)\,\ell}{(k-\ell)\,\delta_1 + \ell\,\delta_3} \right| = \frac{1}{|\delta_1|} \left| \ell - \frac{\delta_3\,\ell^2}{(k-\ell)\,\delta_1 + \ell\,\delta_3} \right| \leq C \, |\ell|^{\gamma_0+2} \,,$$

from which we get

$$\sup_{(\xi,k)\in\mathbb{R}^{d-1}\times\mathbb{Z}} \int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} \left| \frac{\chi_1(\xi,k,\eta,\ell) \, \langle(\xi,k)\rangle^\nu \, \mathbf{K}(k,\ell)}{\langle(\xi-\eta,k-\ell)\rangle^\nu \, \langle(\eta,\ell)\rangle^{\nu_0}} \right|^2 \, {\rm d}\eta \leq C \int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} \frac{|\ell|^{2\,(\gamma_0+2)}}{\langle(\eta,\ell)\rangle^{2\,\nu_0}} \, {\rm d}\eta < +\infty \,,$$

thanks to our choice of ν_0. Applying Lemma 3.3 to the quantity in (3.9), we obtain

$$\int_{\mathbb{R}^{d-1}} \sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} \chi_1(\xi,k,\eta,\ell) \, \langle(\xi,k)\rangle^\nu \, \mathbf{K}(k,\ell) \, \widehat{c_{k-\ell}(u)}(\xi-\eta) \, \widehat{c_\ell(v)}(\eta) \, {\rm d}\eta \right|^2 \, {\rm d}\xi \leq C \, \|u\|_{H^\nu}^2 \, \|v\|_{H^{\nu_0}}^2 \,.$$

Similar arguments yield the bound

$$\int_{\mathbb{R}^{d-1}} \sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{R}^{d-1}} \sum_{\ell\in\mathbb{Z}} \chi_2(\xi,k,\eta,\ell) \, \langle(\xi,k)\rangle^\nu \, \mathbf{K}(k,\ell) \, \widehat{c_{k-\ell}(u)}(\xi-\eta) \, \widehat{c_\ell(v)}(\eta) \, {\rm d}\eta \right|^2 \, {\rm d}\xi \leq C \, \|u\|_{H^{\nu_0}}^2 \, \|v\|_{H^\nu}^2 \,,$$

and the combination of the two previous inequalities gives the expected estimate

$$\int_{\mathbb{R}^{d-1}} \sum_{k\in\mathbb{Z}} \langle(\xi,k)\rangle^{2\,\nu} \, |c_k(\widehat{\mathbb{F}_{\rm per}(\partial_\theta u, \partial_\theta v)})(\xi)|^2 \, {\rm d}\xi \leq C \, \big(\|u\|_{H^{\nu_0}} \, \|v\|_{H^\nu} + \|u\|_{H^\nu} \, \|v\|_{H^{\nu_0}} \big) \,.$$

$\square$

3.3 The iteration scheme

In view of the boundedness property proved in Theorem 3.1, Equation (3.6) is a semilinear perturbation of the Burgers equation (the transport term $\mathbf{w} \cdot \nabla_y$ is harmless), and it is absolutely not surprising that we can solve (3.6) by using the standard energy method with a fixed point iteration. This well-posedness result can be summarized in the following Theorem.

Theorem 3.4. *Let ν_0 be defined as in Theorem 3.1, and let $\nu \geq \nu_0$. Then for all $R > 0$, there exists a time $T > 0$ such that for all data $a_0 \in H^\nu(\mathbb{R}^{d-1} \times (\mathbb{R}/\Theta\,\mathbb{Z}))$ satisfying $\|a_0\|_{H^{\nu_0}} \leq R$, there exists a unique solution $a \in \mathcal{C}([0,T]; H^\nu)$ to the Cauchy problem:*

$$\begin{cases} \partial_t a + \mathbf{w} \cdot \nabla_y a + c\, a\, \partial_\theta a + \mu\, \mathbb{F}_{\mathrm{per}}(\partial_\theta a, \partial_\theta a) = 0\,, \\ a|_{t=0} = a_0\,. \end{cases}$$

In particular, if $a_0 \in H^{+\infty}(\mathbb{R}^{d-1} \times (\mathbb{R}/\Theta\,\mathbb{Z}))$, then $a \in \mathcal{C}([0,T]; H^{+\infty})$ where the time $T > 0$ only depends on $\|a_0\|_{H^{\nu_0}}$.

Proof. The proof follows the standard strategy for quasilinear symmetric systems, see for instance [BGS07, chapter 10] or [Tay97, chapter 16], and we solve the Cauchy problem by the iteration scheme

$$\begin{cases} \partial_t a^{n+1} + \mathbf{w} \cdot \nabla_y a^{n+1} + c\, a^n\, \partial_\theta a^{n+1} + \mu\, \mathbb{F}_{\mathrm{per}}(\partial_\theta a^n, \partial_\theta a^n) = 0\,, \\ a^{n+1}|_{t=0} = a_{0,n+1}\,, \end{cases}$$

where $(a_{0,n})$ is a sequence of, say, Schwartz functions that converges towards a_0 in H^ν, and the scheme is initialized with the choice $a^0 \equiv a_{0,0}$. Given the radius R for the ball in H^{ν_0}, we can choose some time $T > 0$, that only depends on R and ν, such that the sequence (a^n) is bounded in $\mathcal{C}([0,T]; H^\nu)$. The uniform bound in $\mathcal{C}([0,T]; H^\nu)$ is proved by following the exact same ingredients as in the case of the Burgers equation. Contraction in $\mathcal{C}([0,T]; L^2)$ is obtained by computing the equation for the difference $r^{n+1} := a^{n+1} - a^n$, which reads

$$\partial_t r^{n+1} + \mathbf{w} \cdot \nabla_y r^{n+1} + c\, a^n\, \partial_\theta r^{n+1} = -c\, r^n\, \partial_\theta a^n - \mu\, \mathbb{F}_{\mathrm{per}}(\partial_\theta r^n, \partial_\theta a^n) - \mu\, \mathbb{F}_{\mathrm{per}}(\partial_\theta a^{n-1}, \partial_\theta r^n)\,. \tag{3.10}$$

The error terms on the right hand-side are written as

$$\mathbb{F}_{\mathrm{per}}(\partial_\theta r^n, \partial_\theta a^n) = -\frac{2\,\pi}{\Theta\,\delta_1}\, r^n\, \partial_\theta a^n - \frac{\delta_3}{\delta_1}\, \mathbb{F}_{\mathrm{per}}(r^n, \partial^2_{\theta\theta} a^n)\,,$$
$$\mathbb{F}_{\mathrm{per}}(\partial_\theta a^{n-1}, \partial_\theta r^n) = -\frac{2\,\pi}{\Theta\,\delta_3}\, r^n\, \partial_\theta a^{n-1} - \frac{\delta_1}{\delta_3}\, \mathbb{F}_{\mathrm{per}}(\partial^2_{\theta\theta} a^{n-1}, r^n)\,,$$

where we have used (3.5).

The final ingredient in the proof is a continuity estimate of the form

$$\|\mathbb{F}_{\mathrm{per}}(u,v)\|_{L^2} \leq C\, \min\big(\|u\|_{H^{\nu_0-2}}\, \|v\|_{L^2}, \|u\|_{L^2}\, \|v\|_{H^{\nu_0-2}}\big)\,, \tag{3.11}$$

which we now prove for completeness. We apply the Fubini and Parseval-Bessel Theorems to obtain

$$\|\mathbb{F}_{\mathrm{per}}(u,v)\|^2_{L^2} = \Theta \int_{\mathbb{R}^{d-1}} \sum_{k \in \mathbb{Z}} \left| \sum_{\substack{k_1+k_3=k, \\ k_1\, k_3 \neq 0}} \frac{1}{k_1\, \delta_1 + k_3\, \delta_3}\, u_{k_1}\, v_{k_3} \right|^2 \, \mathrm{d}y\,,$$

and then apply the $\ell^1 \star \ell^2 \to \ell^2$ continuity estimate to derive

$$\|\mathbb{F}_{\mathrm{per}}(u,v)\|_{L^2}^2 \le C \int_{\mathbb{R}^{d-1}} \left(\sum_{k\in\mathbb{Z}} |k|^{\gamma_0} |\, |u_k| \right) \sum_{k\in\mathbb{Z}} |v_k|^2 \, \mathrm{d}y \, .$$

We then apply the Cauchy-Schwarz inequality and derive the estimate

$$\|\mathbb{F}_{\mathrm{per}}(u,v)\|_{L^2}^2 \le C \Big(\sup_{y\in\mathbb{R}^{d-1}} \|u(y,\cdot)\|_{H^{\gamma_0+1}(\mathbb{R}/\Theta\,\mathbb{Z})}^2 \Big) \|v\|_{L^2}^2 \, ,$$

which yields (3.11) because the integer ν_0 in Theorem 3.1 can be chosen larger than $(d-1)/2+\gamma_0+3$. (The "symmetric" estimate is obtained by exchanging the roles of u and v.)

At this stage, we multiply Equation (3.10) by r^{n+1} and perform integration by parts to derive

$$\sup_{t\in[0,T]} \|r^{n+1}\|_{L^2}^2 \le \|r^{n+1}|_{t=0}\|_{L^2}^2 + C_0\,T \sup_{t\in[0,T]} \|r^{n+1}\|_{L^2}^2 + C_0\,T \sup_{t\in[0,T]} \|r^{n+1}\|_{L^2} \sup_{t\in[0,T]} \|r^n\|_{L^2} \, ,$$

where the constant C_0 is independent of n and follows from the uniform bound for $\sup_{t\in[0,T]} \|a^n\|_{H^\nu}$. By classical interpolation arguments, (a^n) converges towards a weakly in $L^\infty([0,T];H^\nu)$ and strongly in $\mathcal{C}([0,T];H^{\nu'})$, $\nu'<\nu$. Continuity of a with values in H^ν is recovered by the standard arguments, see, e.g., [Tay97, Proposition 1.4].

If $\nu>\nu_0$, it remains to show that the time T only depends on the norm $\|a_0\|_{H^{\nu_0}}$, and this is where the tame estimate of Theorem 3.1 enters the game. More precisely, we follow the same strategy as in [Tay97, Corollary 1.6], and show that the H^ν-norm of the solution a satisfies a differential inequality of the form

$$\frac{\mathrm{d}\|a(t)\|_{H^\nu}^2}{\mathrm{d}t} \le C_\nu\Big(\|a(t)\|_{H^{\nu_0}}^2\Big)\,\|a(t)\|_{H^\nu}^2 \, ,$$

where C_ν is an increasing function of its argument. In particular, boundedness of $a(t)$ in H^{ν_0} on an interval $[0,T')$, $T'>0$, implies a unique extension of the solution $a\in\mathcal{C}([0,T');H^\nu)$ beyond the time T', which means that the time T of existence for a only depends on $\|a_0\|_{H^{\nu_0}}$. $\square$

3.4 Construction of the leading profile

Theorem 3.4 is the cornerstone of the construction of the leading profile $\mathcal{U}_0$. Solvability of (2.22) for a is summarized in the following result. Recall that the smoothness assumption for G was made in Theorem 1.10.

Corollary 3.5. *There exists $T>0$, and $a\in\mathcal{C}^\infty((-\infty,T];H^{+\infty}(\mathbb{R}^{d-1}\times(\mathbb{R}/\Theta\,\mathbb{Z})))$ solution to (2.22) with $a|_{t<0}=0$. Furthermore, a has mean value zero with respect to the variable θ.*

Proof. Equation (2.22) is easier to solve than the pure Cauchy problem in Theorem 3.4 because we can apply Duhamel's formula starting from the initial condition $a_0=0$. From the assumption of Theorem 1.10, we have $G\in\mathcal{C}^\infty((-\infty,T_0];H^{+\infty}(\mathbb{R}^{d-1}\times(\mathbb{R}/\Theta\,\mathbb{Z})))$ with $T_0>0$ and $G|_{t<0}=0$, so we can find $0<T\le T_0$ and $a\in\mathcal{C}((-\infty,T];H^{+\infty}(\mathbb{R}^{d-1}\times(\mathbb{R}/\Theta\,\mathbb{Z})))$ solution to (2.22) with $a|_{t<0}=0$. Here the time T depends on a fixed norm of G. Then Equation (2.22) yields $a\in\mathcal{C}^\infty((-\infty,T];H^{+\infty}(\mathbb{R}^{d-1}\times(\mathbb{R}/\Theta\,\mathbb{Z})))$ by the standard bootstrap argument and smoothness of G.

For every fixed y, the mean value

$$\underline{a}(t,y) := \frac{1}{\Theta}\int_0^\Theta a(t,y,\theta)\,\mathrm{d}\theta \, ,$$

satisfies the homogeneous transport equation

$$\partial_t \underline{a} + \mathbf{w} \cdot \nabla_y \underline{a} = 0 \,,$$

with zero initial condition, and therefore vanishes. $\square$

After constructing a on the boundary, we can achieve the construction of the leading profile $\mathcal{U}_0$ in the whole domain.

Corollary 3.6. *Up to retricting $T > 0$ in Corollary 3.5, for all $m \in \mathcal{I}$, there exists a unique solution $\sigma_m \in \mathcal{C}^\infty((-\infty, T]; H^{+\infty}(\mathbb{R}^d_+ \times (\mathbb{R}/\Theta\,\mathbb{Z})))$ to (2.12) with $\sigma_m|_{t<0} = 0$ and $\sigma_m|_{x_d=0} = \mathfrak{e}_m\, a$, where the real number $\mathfrak{e}_m$ is defined by $e_m = \mathfrak{e}_m\, r_m$. Furthermore, each σ_m has mean value zero with respect to the variable θ_m.*

The result follows from solving the boundary value problem for the Burgers equation (2.12) with prescribed Dirichlet boundary condition on $\{x_d = 0\}$. This is a (very!) particular case of a quasilinear hyperbolic system with strictly dissipative boundary conditions for which well-posedness follows from the classical theory, see, e.g., [BGS07].

The leading profile $\mathcal{U}_0$ is then given by (2.21) and belongs to $\mathcal{C}^\infty((-\infty, T]; H^{+\infty}(\mathbb{R}^d_+ \times (\mathbb{R}/\Theta\,\mathbb{Z})^N))$. Furthermore, its spectrum with respect to the periodic variables $(\theta_1, \dots, \theta_N)$ is included in the set

$$\mathbb{Z}^N_{\mathcal{I}} := \big\{ \alpha \in \mathbb{Z}^N \, / \, \forall\, m \in \mathcal{O} \,,\, \alpha_m = 0 \big\} \,, \tag{3.12}$$

as claimed in Theorem 1.10.

4 Proof of Theorem 1.10

4.1 The WKB cascade

In this paragraph, we give a more detailed version of (2.6)-(2.8). We plug again the ansatz (2.5) in (1.1) and derive the set of equations (4.1)-(4.3) below. We recall that the operators $\mathcal{L}, \mathcal{M}$ are defined in (2.7), while $L(\partial)$ is defined in (1.4). Then the WKB cascade in the interior reads:

$$\begin{aligned}
&\text{(a)} \quad \mathcal{L}(\partial_\theta)\, \mathcal{U}_0 = 0 \,, \\
&\text{(b)} \quad \mathcal{L}(\partial_\theta)\, \mathcal{U}_1 + L(\partial)\, \mathcal{U}_0 + \mathcal{M}(\mathcal{U}_0, \mathcal{U}_0) = 0 \,, \\
&\text{(c)} \quad \mathcal{L}(\partial_\theta)\, \mathcal{U}_{k+2} + L(\partial)\, \mathcal{U}_{k+1} + \mathcal{M}(\mathcal{U}_0, \mathcal{U}_{k+1}) + \mathcal{M}(\mathcal{U}_{k+1}, \mathcal{U}_0) + \mathbb{F}_k = 0 \,, \quad k \geq 0 \,,
\end{aligned} \tag{4.1}$$

with

$$\begin{aligned}
&\forall\, k \geq 0 \,, \quad \mathbb{F}_k := \partial_j \varphi_m \left(\sum_{\ell=2}^{k+2} \frac{1}{\ell!} \sum_{\kappa_1 + \cdots + \kappa_\ell = k+2-\ell} \mathrm{d}^\ell A_j(0) \cdot (\mathcal{U}_{\kappa_1}, \dots, \mathcal{U}_{\kappa_\ell}) \right) \partial_{\theta_m} \mathcal{U}_0 \\
&\qquad\qquad + \sum_{\ell=1}^{k+1} \mathbb{A}_j^{k+2-\ell}\, \partial_j \mathcal{U}_{\ell-1} + \partial_j \varphi_m \sum_{\ell=1}^{k} \mathbb{A}_j^{k+2-\ell}\, \partial_{\theta_m} \mathcal{U}_\ell \,, \\
&\forall\, \nu \geq 1 \,, \quad \mathbb{A}_j^\nu := \sum_{\ell=1}^{\nu} \frac{1}{\ell!} \sum_{\kappa_1 + \cdots + \kappa_\ell = \nu - \ell} \mathrm{d}^\ell A_j(0) \cdot (\mathcal{U}_{\kappa_1}, \dots, \mathcal{U}_{\kappa_\ell}) \,,
\end{aligned} \tag{4.2}$$

Observe that (4.1)(c) coincides with (2.6)(c) for $k = 0$. Furthermore, each matrix $\mathbb{A}_j^\nu$, $\nu \geq 1$, only depends on $\mathcal{U}_0, \dots, \mathcal{U}_{\nu-1}$, and therefore each source term $\mathbb{F}_k$, $k \geq 0$, only depends on $\mathcal{U}_0, \dots, \mathcal{U}_k$.

The set of boundary conditions for (4.1) reads (recall $B = \mathrm{d}b(0)$):

$$
\begin{aligned}
&\text{(a)} \quad B\,\mathcal{U}_0 = 0\,, \\
&\text{(b)} \quad B\,\mathcal{U}_1 + \frac{1}{2}\,\mathrm{d}^2 b(0) \cdot (\mathcal{U}_0, \mathcal{U}_0) = G(t, y, \theta_0)\,, \\
&\text{(c)} \quad B\,\mathcal{U}_{k+2} + \mathrm{d}^2 b(0) \cdot (\mathcal{U}_0, \mathcal{U}_{k+1}) + \mathbb{G}_k = 0\,, \quad k \geq 0\,,
\end{aligned}
\tag{4.3}
$$

with

$$
\forall\, k \geq 0\,, \quad \mathbb{G}_k := \sum_{\ell=3}^{k+3} \frac{1}{\ell!} \sum_{\kappa_1+\cdots+\kappa_\ell = k+3-\ell} \mathrm{d}^\ell b(0) \cdot (\mathcal{U}_{\kappa_1}, \dots, \mathcal{U}_{\kappa_\ell})\,.
\tag{4.4}
$$

In (4.3) and (4.4), all functions on the left hand side are evaluated at $x_d = 0, \theta_1 = \cdots = \theta_N = \theta_0$. The source term $\mathbb{G}_k$ in (4.3)(c) only depends on $\mathcal{U}_0, \dots, \mathcal{U}_k$.

We are looking for a sequence of profiles $(\mathcal{U}_k)_{k\in\mathbb{N}}$ that satisfies (4.1)-(4.3), and $\mathcal{U}_k|_{t<0} = 0$ for all k.

4.2 Construction of correctors

Some notation will be useful in the arguments below. For any function f that depends on $(t, x, \theta_1, \dots, \theta_N)$, with Θ-periodicity with respect to each θ_m, we decompose f as

$$
f = \underline{f}(t, x) + \sum_{m=1}^{N} f^m(t, x, \theta_m) + f^{\mathrm{nc}}(t, x, \theta_1, \dots, \theta_N)\,,
$$

where $\underline{f}$ stands for the mean value of f on the torus $(\mathbb{R}/\Theta\,\mathbb{Z})^N$, each f^m incorporates the θ_m-modes of f (in particular, the spectrum of f^m is included in $\mathbb{Z}^{N;1}$ and f_m has mean zero with respect to θ_m), and the spectrum of f^{nc} is included in $\mathbb{Z}^N \setminus \mathbb{Z}^{N;1}$. Here, the spectrum only refers to the Fourier decomposition of f with respect to $(\theta_1, \dots, \theta_N)$. The mappings $f \mapsto f^m$ and $f \mapsto f^{\mathrm{nc}}$ are continuous on $\mathcal{C}^\infty((-\infty, T]; H^{+\infty}(\mathbb{R}^d_+ \times (\mathbb{R}/\Theta\,\mathbb{Z})^N))$. Furthermore, if the spectrum of f is included in $\mathbb{Z}^N_{\mathcal{I}}$, then f^{nc} belongs to the space of profiles $\mathbb{P}^{\mathrm{nc}}$ defined in Lemma 4.1 below.

The following observation is well-known in the theory of geometric optics, see for instance [JMR93, Wil99], and relies on Assumption 1.9.

Lemma 4.1. *The operator $\mathcal{L}(\partial_\theta)$ is a bounded isomorphism from $\mathbb{P}^{\mathrm{nc}}$ into itself, where*

$$
\mathbb{P}^{\mathrm{nc}} := \Big\{ f \in \mathcal{C}^\infty((-\infty, T]; H^{+\infty}(\mathbb{R}^d_+ \times (\mathbb{R}/\Theta\,\mathbb{Z})^N)) \,/\, \mathrm{Spectrum}\ (f) \subset \mathbb{Z}^N_{\mathcal{I}} \setminus \mathbb{Z}^{N;1} \Big\}\,.
$$

Indeed, for $\alpha \in \mathbb{Z}^N_{\mathcal{I}} \setminus \mathbb{Z}^{N;1}$, the matrix $L(\mathrm{d}(\alpha \cdot \Phi))$ is invertible and the norm of its inverse is bounded polynomially in $|\alpha|$ (the degree of the polynomial being fixed). We shall feel free to write $\mathcal{L}(\partial_\theta)^{-1} f^{\mathrm{nc}}$ when f^{nc} is an element of $\mathbb{P}^{\mathrm{nc}}$.

Unsurprisingly, the construction of the sequence $(\mathcal{U}_k)$ is based on an induction process. We formulate our induction assumption.

($\mathbf{H}_n$) There exist profiles $\mathcal{U}_0, \dots, \mathcal{U}_n$ in $\mathcal{C}^\infty((-\infty, T]; H^{+\infty}(\mathbb{R}^d_+ \times (\mathbb{R}/\Theta\,\mathbb{Z})^N))$ that vanish for $t < 0$, whose $(\theta_1, \dots, \theta_N)$-spectrum is included in $\mathbb{Z}^N_{\mathcal{I}}$, and that satisfies

- (4.1)(a) and (4.3)(a) if $n=0$,
- (4.1)(a)-(b) and (4.3)(a)-(b) if $n=1$,
- (4.1)(a)-(b), (4.3)(a)-(b), (4.1)(c) and (4.3)(c) up to order $n-2$ if $n\geq 2$,
- Compatibility condition in the interior:

$$\forall\, m\in\mathcal{I}\,,\quad \ell_m\, F_n^m=0\,,\quad \text{and } \underline{F_n}=0\,,\tag{4.5}$$

- Compatibility condition on the boundary:

$$\underline{b}\Big(-\sum_{m\in\mathcal{I}} B\,R_m\,F_n^m|_{x_d=0,\theta_m=\theta_0}-B\,\partial_{\theta_0}((\mathcal{L}(\partial_\theta)^{-1}\,F_n^{\text{nc}})|_{x_d=0,\theta_1=\cdots=\theta_N=\theta_0})\\ +\partial_{\theta_0}G_n(t,y,\theta_0)\Big)=0\,,\tag{4.6}$$

where in (4.5) and (4.6), we have set:

$$F_n:=\begin{cases} L(\partial)\,\mathcal{U}_0+\mathcal{M}(\mathcal{U}_0,\mathcal{U}_0)\,, & \text{if } n=0\,,\\ L(\partial)\,\mathcal{U}_n+\mathcal{M}(\mathcal{U}_0,\mathcal{U}_n)+\mathcal{M}(\mathcal{U}_n,\mathcal{U}_0)+\mathbb{F}_{n-1}\,, & \text{if } n\geq 1\,,\end{cases}\tag{4.7}$$

$$G_n:=\begin{cases} \dfrac{1}{2}\,\mathrm{d}^2b(0)\cdot(\mathcal{U}_0,\mathcal{U}_0)|_{x_d=0,\theta_1=\cdots=\theta_N=\theta_0}-G(t,y,\theta_0)\,, & \text{if } n=0\,,\\ \mathrm{d}^2b(0)\cdot(\mathcal{U}_0,\mathcal{U}_n)|_{x_d=0,\theta_1=\cdots=\theta_N=\theta_0}+\mathbb{G}_{n-1}\,, & \text{if } n\geq 1\,.\end{cases}\tag{4.8}$$

Recall that $\mathbb{F}_k$ and $\mathbb{G}_k$ are defined in (4.2) and (4.4), so the above source terms F_n, G_n only depend on $\mathcal{U}_0,\dots,\mathcal{U}_n$. Several properties of these source terms are made precise below which, in particular, will justify why we can apply the operator $\mathcal{L}(\partial_\theta)^{-1}$ to F_n^{nc}.

The reader can verify that our construction of the leading profile $\mathcal{U}_0$ in Sections 2 and 3 proves that $(\mathbf{H}_0)$ holds. In that case, (4.6) reduces to (2.22), which was solved in Section 3. The compatibility conditions (4.5) in the interior give the decoupled Burgers equations (2.12) for each incoming amplitude σ_m.

Our goal is to show that $(\mathbf{H}_n)$ implies $(\mathbf{H}_{n+1})$, which will imply that there exists a sequence of profiles $(\mathcal{U}_k)_{k\in\mathbb{N}}$ in $\mathcal{C}^\infty((-\infty,T];H^{+\infty}(\mathbb{R}^d_+\times(\mathbb{R}/\Theta\,\mathbb{Z})^N))$ that satisfies (4.1)-(4.3), and $\mathcal{U}_k|_{t<0}=0$ for all k. We decompose the analysis in several steps, as in Section 2, assuming from now on that $(\mathbf{H}_n)$ holds for some integer $n\in\mathbb{N}$.

Step 1: properties of F_n, G_n, and definition of $\mathcal{U}_{n+1}^{\text{nc}}$, $(I-P_m)\,\mathcal{U}_{n+1}^m$.

From assumption $(\mathbf{H}_n)$, the profiles $\mathcal{U}_0,\dots,\mathcal{U}_n$ belong to $\mathcal{C}^\infty((-\infty,T];H^{+\infty}(\mathbb{R}^d_+\times(\mathbb{R}/\Theta\,\mathbb{Z})^N))$, vanish for $t<0$, and their spectrum is a subset of $\mathbb{Z}^N_{\mathcal{I}}$. The space of such functions is an algebra, and therefore, we can verify from (4.2) and (4.7) that F_n belongs to $\mathcal{C}^\infty((-\infty,T];H^{+\infty}(\mathbb{R}^d_+\times(\mathbb{R}/\Theta\,\mathbb{Z})^N))$, vanishes for $t<0$, and its spectrum is a subset of $\mathbb{Z}^N_{\mathcal{I}}$. Consequently, F_n^{nc} belongs to the space of profiles $\mathbb{P}^{\text{nc}}$ defined in Lemma 4.1.

Our goal is to construct a profile $\mathcal{U}_{n+1}$ that satisfies

$$\mathcal{L}(\partial_\theta)\,\mathcal{U}_{n+1}+F_n=0\,.\tag{4.9}$$

We first observe that if $\alpha\in\mathbb{Z}^N$ is a noncharacteristic mode, that is, $\alpha\in\mathbb{Z}^N\setminus\mathbb{Z}^{N;1}$, and if moreover α has one nonzero coordinate α_m with $m\in\mathcal{O}$, then the α-Fourier coefficient of $\mathcal{U}_{n+1}$ vanishes. This implies that any solution to (4.9) has its spectrum included in $\mathbb{Z}^N_{\mathcal{I}}$. We thus define $\mathcal{U}^{\rm nc}_{n+1} := -\mathcal{L}(\partial_\theta)^{-1}\, F^{\rm nc}_n$, so $\mathcal{U}^{\rm nc}_{n+1}$ belongs to $\mathbb{P}^{\rm nc}$ and vanishes for $t<0$.

For all $m=1,\dots,N$, we define $(I-P_m)\,\mathcal{U}^m_{n+1}$ as the unique mean zero solution to

$$(I-P_m)\,\partial_{\theta_m}\mathcal{U}^m_{n+1} = -R_m\, F^m_n\,.$$

In particular, we can write $\mathcal{U}^m_{n+1} = \sigma^m_{n+1}\, r_m$ for $m\in\mathcal{O}$ since F_n has no outgoing mode. Due to the compatibility condition (4.5) for $m\in\mathcal{I}$, we have

$$\forall\, m=1,\dots,N\,,\quad L({\rm d}\varphi_m)\,(I-P_m)\,\partial_{\theta_m}\mathcal{U}^m_{n+1}+F^m_n=0\,,$$

which means that the components of $\mathcal{U}_{n+1}$ that we have already defined satisfy

$$\mathcal{L}(\partial_\theta)\,\left(\sum_{m=1}^N (I-P_m)\,\mathcal{U}^m_{n+1}+\mathcal{U}^{\rm nc}_{n+1}\right)+F_n=0\,,$$

because F_n has zero mean value. It is clear that the nonpolarized components $(I-P_m)\,\mathcal{U}^m_{n+1}$ belong to $\mathcal{C}^\infty((-\infty,T];H^{+\infty}(\mathbb{R}^d_+\times(\mathbb{R}/\Theta\,\mathbb{Z})^N))$ and vanish for $t<0$, because the F^m_n's do so.

Step 2: $\mathcal{U}_{n+1}$ has no outgoing mode.

Let $m\in\mathcal{O}$. We focus on (4.1)(c) for $k=n$. Since $\mathbb{F}_n$ has no outgoing mode, the profile $\mathcal{U}_{n+1}$ must necessarily satisfy

$$\ell_m\, L(\partial)\,(\sigma^m_{n+1}\, r_m)+\ell_m\,(\mathcal{M}(\mathcal{U}_0,\mathcal{U}_{n+1})+\mathcal{M}(\mathcal{U}_{n+1},\mathcal{U}_0))^m=0\,.$$

However, the leading profile $\mathcal{U}_0$ is given in (2.21), and since the only noncharacteristic modes of $\mathcal{U}_{n+1}$ belong to $\mathbb{Z}^N_{\mathcal{I}}$, we observe that it is not possible to generate a θ_m-mode in $\mathcal{M}(\mathcal{U}_0,\mathcal{U}_{n+1})$ nor in $\mathcal{M}(\mathcal{U}_{n+1},\mathcal{U}_0)$. This means that the amplitude σ^m_{n+1} satisfies the outgoing transport equation

$$\partial_t\sigma^m_{n+1}+\mathbf{v}_m\cdot\nabla_x\sigma^m_{n+1}=0\,,$$

and therefore vanishes.

Since the profile $\mathcal{U}_{n+1}$ has no outgoing mode, it reads

$$\mathcal{U}_{n+1}=\underline{\mathcal{U}_{n+1}}+\sum_{m\in\mathcal{I}}P_m\,\mathcal{U}^m_{n+1}+\sum_{m\in\mathcal{I}}(I-P_m)\,\mathcal{U}^m_{n+1}+\mathcal{U}^{\rm nc}_{n+1}\,,$$

and it only remains to determine the mean value $\underline{\mathcal{U}_{n+1}}$ and the polarized components $P_m\,\mathcal{U}^m_{n+1}$, $m\in\mathcal{I}$. Let us observe that such components have no influence on the fulfillment of (4.9), no matter how we define them because they belong to the kernel of $\mathcal{L}(\partial_\theta)$. Hence we shall no longer focus on (4.9), but rather on the boundary conditions for $\mathcal{U}_{n+1}$ and the interior compatibility condition at the next order.

Step 3: determining $\underline{\mathcal{U}_{n+1}}$.

Let us first derive the interior equation for $\underline{\mathcal{U}_{n+1}}$. We introduce the notation $P_m\,\mathcal{U}^m_{n+1}=\sigma^m_{n+1}\,r_m$ for $m\in\mathcal{I}$. We consider (4.1)(c) for $k=n$, and take its mean value on the torus, observing first that both terms $\mathcal{M}(\mathcal{U}_0,\mathcal{U}^{\rm nc}_{n+1})$ and $\mathcal{M}(\mathcal{U}^{\rm nc}_{n+1},\mathcal{U}_0)$ have zero mean value. Hence we derive the equation

$$\begin{aligned}L(\partial)\,\underline{\mathcal{U}_{n+1}}&+\sum_{m\in\mathcal{I}}\int_0^\Theta \sigma_m\,\partial_{\theta_m}\sigma^m_{n+1}\,\frac{{\rm d}\theta_m}{\Theta}\,\partial_j\varphi_m\,({\rm d}A_j(0)\cdot r_m)\,r_m\\&+\sum_{m\in\mathcal{I}}\int_0^\Theta \sigma^m_{n+1}\,\partial_{\theta_m}\sigma_m\,\frac{{\rm d}\theta_m}{\Theta}\,\partial_j\varphi_m\,({\rm d}A_j(0)\cdot r_m)\,r_m+\mathcal{F}_n(t,x)=0\,,\end{aligned}$$

where the source term $\mathcal{F}_n$ is defined by:

$$\mathcal{F}_n := \underline{\mathbb{F}_n} + \sum_{m \in \mathcal{I}} \underline{\mathcal{M}(\mathcal{U}_0, (I - P_m) \mathcal{U}_{n+1}^m)} + \sum_{m \in \mathcal{I}} \underline{\mathcal{M}((I - P_m) \mathcal{U}_{n+1}^m, \mathcal{U}_0)} .$$

We observe that each sum of integrals

$$\int_0^\Theta \sigma_m \, \partial_{\theta_m} \sigma_{n+1}^m \, \frac{\mathrm{d}\theta_m}{\Theta} + \int_0^\Theta \sigma_{n+1}^m \, \partial_{\theta_m} \sigma_m \, \frac{\mathrm{d}\theta_m}{\Theta}$$

vanishes, and therefore $\underline{\mathcal{U}_{n+1}}$ must satisfy the system

$$L(\partial) \, \underline{\mathcal{U}_{n+1}} = -\mathcal{F}_n \, ,$$

in $(-\infty, T] \times \mathbb{R}^d_+$. The source term $\mathcal{F}_n$ belongs to $\mathcal{C}^\infty((-\infty, T]; H^{+\infty}(\mathbb{R}^d_+))$ and vanishes for $t < 0$.

The boundary conditions for $\underline{\mathcal{U}_{n+1}}$ are obtained by taking the mean value of (4.3)(b) if $n = 0$ or (4.3)(c) for $k = n-1$ if $n \geq 1$. In any case we find that $\underline{\mathcal{U}_{n+1}}|_{x_d=0}$ must satisfy

$$B \, \underline{\mathcal{U}_{n+1}}|_{x_d=0} + B \, \underline{(\mathcal{U}_{n+1}^{\mathrm{nc}}|_{x_d=0, \theta_1 = \cdots = \theta_N = \theta_0})} + \underline{G_n}(t, y) = 0 \, .$$

Since $\mathcal{U}_{n+1}^{\mathrm{nc}}$ has already been determined, we can apply the well-posedness result of [Cou05], supplemented with the regularity result in [MS11], and construct a solution $\underline{\mathcal{U}_{n+1}} \in \mathcal{C}^\infty((-\infty, T]; H^{+\infty}(\mathbb{R}^d_+))$ to the above equations (in the interior and on the boundary).

Step 4: determining $P_m \, \mathcal{U}_{n+1}^m$. Part I.

We keep the notation $P_m \, \mathcal{U}_{n+1}^m = \sigma_{n+1}^m \, r_m$ for $m \in \mathcal{I}$. The evolution of σ_{n+1}^m is obtained by imposing the compatibility condition:

$$\ell_m \, L(\partial) \, \mathcal{U}_{n+1}^m + \ell_m \, \Big(\mathcal{M}(\mathcal{U}_0, \mathcal{U}_{n+1}) + \mathcal{M}(\mathcal{U}_{n+1}, \mathcal{U}_0) \Big)^m = -\ell_m \, \mathbb{F}_n^m \, .$$

Keeping on the left hand side only what is still unknown, we end up with:

$$\begin{aligned} &\partial_t \sigma_{n+1}^m + \mathbf{v}_m \cdot \nabla_x \sigma_{n+1}^m + c_m \left(\sigma_m \, \partial_{\theta_m} \, \sigma_{n+1}^m + \sigma_{n+1}^m \, \partial_{\theta_m} \, \sigma_m \right) \\ &\qquad = -\frac{\ell_m}{\ell_m \, r_m} \Big[\mathbb{F}_n^m + L(\partial) \, (I - P_m) \, \mathcal{U}_{n+1}^m + \mathcal{M}(\underline{\mathcal{U}_{n+1}}, \sigma_m \, r_m) \\ &\quad + \mathcal{M}(\sigma_m \, r_m, (I - P_m) \, \mathcal{U}_{n+1}^m) + \mathcal{M}((I - P_m) \, \mathcal{U}_{n+1}^m, \sigma_m \, r_m) + \Big(\mathcal{M}(\mathcal{U}_0, \mathcal{U}_{n+1}^{\mathrm{nc}}) + \mathcal{M}(\mathcal{U}_{n+1}^{\mathrm{nc}}, \mathcal{U}_0) \Big)^m \Big] \, , \end{aligned}$$

where the constant c_m is the one defined in (2.12). We have thus found that the σ_{n+1}^m's must satisfy decoupled incoming transport equations in $\mathbb{R}^d_+ \times (\mathbb{R}/\Theta \, \mathbb{Z})$ with infinitely smooth coefficients and source terms (all vanishing for $t < 0$). The only task left is therefore to determine the trace of each σ_{n+1}^m, $m \in \mathcal{I}$, on $x_d = 0$.

We recall the decomposition (2.4) introduced in Section 2. We thus introduce a decomposition

$$\sum_{m \in \mathcal{I}} \sigma_{n+1}^m(t, y, 0, \theta_0) \, r_m = a_{n+1}(t, y, \theta_0) \, e + \check{\mathcal{U}}_{n+1}(t, y, \theta_0) \, , \tag{4.10}$$

where a_{n+1} is an unknown scalar function (with zero mean value with respect to θ_0), and $\check{\mathcal{U}}_{n+1}$ takes its values in $\check{\mathbb{E}}^s(\underline{\tau},\underline{\eta})$. Thanks to the compatibility condition (4.6), the function $\check{\mathcal{U}}_{n+1}$ is uniquely determined by solving

$$
\begin{aligned}
&B\,\check{\mathcal{U}}_{n+1} - \partial_{\theta_0}^{-1}\left(\sum_{m\in\mathcal{I}} B\,R_m\,F_n^m|_{x_d=0,\theta_m=\theta_0}\right) \\
&\quad - B\left((\mathcal{L}(\partial_\theta)^{-1}\,F_n^{\rm nc})|_{x_d=0,\theta_1=\cdots=\theta_N=\theta_0} - \underline{(\mathcal{L}(\partial_\theta)^{-1}\,F_n^{\rm nc})|_{x_d=0,\theta_1=\cdots=\theta_N=\theta_0}}\right) + (G_n-\underline{G_n})(t,y,\theta_0)=0\,,
\end{aligned}
$$

where $\partial_{\theta_0}^{-1}$ denotes the inverse of ∂_{θ_0} when restricted to zero mean value functions. We get $\check{\mathcal{U}}_{n+1}\in\mathcal{C}^\infty((-\infty,T];H^{+\infty}(\mathbb{R}^{d-1}\times(\mathbb{R}/\Theta\,\mathbb{Z})))$, and $\check{\mathcal{U}}_{n+1}|_{t<0}=0$.

Before going on, we observe that no matter how we define the scalar function a_{n+1}, our definitions so far of all components of $\mathcal{U}_{n+1}$ give the relation

$$
B\,\mathcal{U}_{n+1}|_{x_d=0,\theta_1=\cdots=\theta_N=\theta_0}+G_n=0\,, \tag{4.11}
$$

which means that (4.3)(b) will be satisfied if $n=0$, or (4.3)(c) will be satisfied for $k=n-1$ if $n\geq 1$. At this stage, it only remains to determine the scalar function a_{n+1}, and the trace of σ_{n+1}^m on $\{x_d=0\}$ will be obtained by computing

$$
\sigma_{n+1}^m(t,y,0,\theta_0)\,r_m=a_{n+1}(t,y,\theta_0)\,e_m+\check{\mathcal{U}}_{n+1,m}(t,y,\theta_0)\,,
$$

where $\check{\mathcal{U}}_{n+1,m}$ stands for the component of $\check{\mathcal{U}}_{n+1}$ on r_m in the decomposition (2.1).

Step 5: determining $P_m\,\mathcal{U}_{n+1}^m$. Part II. Evolution equation for a_{n+1}.

This step mimics the analysis in Section 2, and more specifically Steps 4 and 5 there. Let us assume that $\mathcal{U}_{n+2}$ has no outgoing mode, which will be fully justified at the next order of the induction process. The nonpolarized components of the corrector $\mathcal{U}_{n+2}^m$ are obtained by looking at (4.1)(c) for $k=n$. For $m\in\mathcal{I}$, we thus define $(I-P_m)\,\mathcal{U}_{n+2}^m$ as the solution to

$$
\begin{aligned}
&(I-P_m)\,\partial_{\theta_m}\mathcal{U}_{n+2}^m+R_m\,L(\partial)(\sigma_{n+1}^m\,r_m)+\big(\sigma_m\,\partial_{\theta_m}\sigma_{n+1}^m+\sigma_{n+1}^m\,\partial_{\theta_m}\sigma_m\big)\,\partial_j\varphi_m\,R_m\,({\rm d}A_j(0)\cdot r_m)\,r_m \\
&\qquad =-R_m\,\mathbb{F}_n^m-R_m\,L(\partial)(I-P_m)\,\mathcal{U}_{n+1}^m-\mathcal{M}(\underline{\mathcal{U}_{n+1}},\sigma_m\,r_m) \\
&\quad -\mathcal{M}(\sigma_m\,r_m,(I-P_m)\,\mathcal{U}_{n+1}^m)-\mathcal{M}((I-P_m)\,\mathcal{U}_{n+1}^m,\sigma_m\,r_m)-\Big(\mathcal{M}(\mathcal{U}_0,\mathcal{U}_{n+1}^{\rm nc})+\mathcal{M}(\mathcal{U}_{n+1}^{\rm nc},\mathcal{U}_0)\Big)^m\,,
\end{aligned}
$$

where we should keep in mind that σ_{n+1}^m is still not fully determined, but the right hand side of the equality has already been constructed. Since the partial inverses R_m satisfy $R_m\,A_d(0)\,P_m=0$, the term $R_m\,L(\partial)(\sigma_{n+1}^m\,r_m)$ only involves tangential differentiation with respect to the boundary $\{x_d=0\}$, so we can take the trace of the latter equation on the boundary. We then substitute $a_{n+1}(t,y,\theta_m)\,e_m+\check{\mathcal{U}}_{n+1,m}(t,y,\theta_m)$ for $\sigma_{n+1}^m|_{x_d=0}\,r_m$. These operations yield

$$
\begin{aligned}
&\underline{b}\,B\sum_{m\in\mathcal{I}}(I-P_m)\,(\partial_{\theta_m}\mathcal{U}_{n+2}^m)|_{x_d=0,\theta_m=\theta_0} \\
&\qquad =-X_{\rm Lop}a_{n+1}+(2\,\upsilon-\underline{b}\,{\rm d}^2b(0)\cdot(e,e))\,\big(a\,\partial_{\theta_0}a_{n+1}+a_{n+1}\,\partial_{\theta_0}a\big)+g_{1,n}(t,y,\theta_0)\,,
\end{aligned} \tag{4.12}
$$

with υ and $X_{\rm Lop}$ defined in (2.23), and $g_{1,n}$ is explicitly computable from all previously determined quantities.

We now determine $\mathcal{U}_{n+2}^{\mathrm{nc}}$ by using (4.1)(c) for $k=n$, computing the noncharacteristic components and by taking the trace on $x_d=0$. All these operations lead to the equation

$$\begin{aligned}\mathcal{L}(\partial_\theta)\,\mathcal{U}_{n+2}^{\mathrm{nc}}|_{x_d=0} = &-\sum_{m\in\mathcal{I}}\Big[\mathcal{M}(\mathcal{U}_0,P_m\,\mathcal{U}_{n+1}^m)+\mathcal{M}(P_m\,\mathcal{U}_{n+1}^m,\mathcal{U}_0)\Big]^{\mathrm{nc}}|_{x_d=0}\\ &-\sum_{m\in\mathcal{I}}\Big[\mathcal{M}(\mathcal{U}_0,(I-P_m)\,\mathcal{U}_{n+1}^m)+\mathcal{M}((I-P_m)\,\mathcal{U}_{n+1}^m,\mathcal{U}_0)\Big]^{\mathrm{nc}}|_{x_d=0}\\ &-\Big[\mathcal{M}(\mathcal{U}_0,\mathcal{U}_{n+1}^{\mathrm{nc}})+\mathcal{M}(\mathcal{U}_{n+1}^{\mathrm{nc}},\mathcal{U}_0)\Big]^{\mathrm{nc}}|_{x_d=0}-\mathbb{F}_n^{\mathrm{nc}}|_{x_d=0}-L(\partial)\,\mathcal{U}_{n+1}^{\mathrm{nc}}|_{x_d=0}\,.\end{aligned}$$

We then use the decomposition:

$$P_m\,\mathcal{U}_{n+1}^m|_{x_d=0}=a_{n+1}(t,y,\theta_m)\,e_m+\check{\mathcal{U}}_{n+1,m}(t,y,\theta_m)\,,$$

where $\check{\mathcal{U}}_{n+1,m}$ has already been determined, and we thus obtain the expression:

$$\mathcal{U}_{n+2}^{\mathrm{nc}}|_{x_d=0,\theta_1=\cdots=\theta_N=\theta_0}=\mathbb{B}_{\mathrm{per}}(a,a_{n+1})+g_{2,n}\,, \tag{4.13}$$

where

$$\mathbb{B}_{\mathrm{per}}[u,v]:=-\sum_{\substack{m_1\neq m_2\\ m_1,m_2\in\mathcal{I}}}\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{k_{m_1}+k_{m_2}=k,\\ k_{m_1}\,k_{m_2}\neq 0}}(u_{k_{m_1}}\,v_{k_{m_2}}+u_{k_{m_2}}\,v_{k_{m_1}})\,L(k_{m_1}\,\mathrm{d}\varphi_{m_1}+k_{m_2}\,\mathrm{d}\varphi_{m_2})^{-1}\,(k_{m_2}\,E_{m_2,m_1})\right)\mathrm{e}^{2\,i\,\pi\,k\,\theta_0/\Theta}\,,$$

the vectors E_{m_1,m_2} are defined in (2.25), and $g_{2,n}$ is explicitly computable from all previously determined quantities.

We now consider (4.3)(c) for $k=n$, which we rewrite equivalently as

$$\begin{aligned}&B\,\underline{\mathcal{U}_{n+2}}|_{x_d=0}+B\sum_{m\in\mathcal{I}}P_m\,\mathcal{U}_{n+2}^m|_{x_d=0,\theta_m=\theta_0}\\ &+B\sum_{m\in\mathcal{I}}(I-P_m)\,\mathcal{U}_{n+2}^m|_{x_d=0,\theta_m=\theta_0}+B\,\mathcal{U}_{n+2}^{\mathrm{nc}}|_{x_d=0,\theta_1=\cdots=\theta_N=\theta_0}+a\,a_{n+1}\,\mathrm{d}^2b(0)\cdot(e,e)\\ &+\mathrm{d}^2b(0)\cdot\left(a\,e,\underline{\mathcal{U}_{n+1}}+\check{\mathcal{U}}_{n+1}+\sum_{m\in\mathcal{I}}(I-P_m)\,\mathcal{U}_{n+1}^m+\mathcal{U}_{n+1}^{\mathrm{nc}}\right)+\mathbb{G}_n=0\,.\end{aligned} \tag{4.14}$$

We differentiate the latter equation with respect to θ_0, apply the row vector $\underline{b}$ and use (4.12) and (4.13) to derive the governing equation for a_{n+1}:

$$2\,v\,\big(a\,\partial_{\theta_0}a_{n+1}+a_{n+1}\,\partial_{\theta_0}a\big)-X_{\mathrm{Lop}}a_{n+1}+\partial_{\theta_0}\,Q_{\mathrm{per}}[a,a_{n+1}]+\partial_{\theta_0}\,Q_{\mathrm{per}}[a_{n+1},a]=g_n\,, \tag{4.15}$$

where g_n incorporates all contributions from the source terms $g_{1,n},g_{2,n}$ and the one obtained after differentiating the last line of (4.14) and applying $\underline{b}$. Moreover, Q_{per} is the operator defined in (2.24).

Observe that the above governing equation (4.15) for a_{n+1} is a linearized version of (2.22). Well-posedness for (4.15) follows from the same arguments as those we have used in Section 3, namely from

Theorem 3.1 which shows that (4.15) is a transport equation for a_{n+1} that is perturbed by a nonlocal zero order term. We thus construct a solution $a_{n+1} \in \mathcal{C}^\infty((-\infty,T];H^{+\infty}(\mathbb{R}^{d-1} \times (\mathbb{R}/\Theta\,\mathbb{Z})))$ to (4.15) that vanishes for $t<0$.

Step 6: conclusion.

We have now determined each component of $\mathcal{U}_{n+1}$, which gives a profile in $\mathcal{C}^\infty((-\infty,T];H^{+\infty}(\mathbb{R}^d_+ \times (\mathbb{R}/\Theta\,\mathbb{Z})^N))$ with its spectrum in $\mathbb{Z}^N_{\mathcal{I}}$. Moreover, $\mathcal{U}_{n+1}$ vanishes for $t<0$, and it satisfies (4.9), (4.11). It is also a simple exercise to verify that, if we define F_{n+1} according to (4.7), our construction of $\mathcal{U}_{n+1}$ gives the compatibility conditions

$$\forall\, m \in \mathcal{I}\,, \quad \ell_m\, F^m_{n+1} = 0\,, \quad \text{and} \quad \underline{F_{n+1}} = 0\,,$$

which is nothing but (4.5) at the order $n+1$. Step 5 above also shows that, with G_{n+1} defined as in (4.8), we have the compatibility condition:

$$\underline{b}\Big(-\sum_{m\in\mathcal{I}} B\,R_m\,F^m_{n+1}|_{x_d=0,\theta_m=\theta_0} - B\,\partial_{\theta_0}((\mathcal{L}(\partial_\theta)^{-1}\,F^{\rm nc}_{n+1})|_{x_d=0,\theta_1=\cdots=\theta_N=\theta_0}) + \partial_{\theta_0} G_{n+1}(t,y,\theta_0)\Big) = 0\,,$$

which is nothing but (4.6) at the order $n+1$. We have therefore proved that $(\mathbf{H}_{n+1})$ holds, which completes the induction.

4.3 Proof of Theorem 1.10

We now quickly complete the proof of Theorem 1.10. The analysis in Paragraph 4.2 shows that there exists a sequence of profiles $(\mathcal{U}_n)_{n\ge 0}$ in $\mathcal{C}^\infty((-\infty,T];H^{+\infty}(\mathbb{R}^d_+ \times (\mathbb{R}/(\Theta\,\mathbb{Z}))^N))$ that satisfies the WKB cascade (4.1), (4.3), and $\mathcal{U}_n|_{t<0} = 0$ for all $n \in \mathbb{N}$. Moreover, each profile $\mathcal{U}_n$ has its θ-spectrum included in $\mathbb{Z}^N_{\mathcal{I}}$. The uniqueness of such a sequence also follows from an induction argument, where we use the regularity of each profile to justify all computations that we have made in order to construct the $\mathcal{U}_n$'s (successive differentiations, identification of Fourier coefficients, substitution $(\theta_1,\dots,\theta_N) \to \theta_0$ on the boundary etc.). Uniqueness of the sequence $(\mathcal{U}_n)_{n\ge0}$ then follows from the uniqueness of smooth solutions to all amplitude equations we have had to solve: Equation (2.22) and its linearized version on the boundary, Burgers equation (2.12) and its linearized version in the interior. All other operations, such as the determination of the non-characteristic components, are "algebraic" and obviously admit a single smooth solution.

To complete the proof of Theorem 1.10, we thus only need to show that the approximate solutions built from the sequence $(\mathcal{U}_n)_{n\ge0}$ satisfy the error estimates claimed in Theorem 1.10. We thus consider

$$u^{app,N_1,N_2}_\varepsilon(t,x) := \sum_{n=0}^{N_1+N_2} \varepsilon^{1+n}\,\mathcal{U}_n\left(t,x,\frac{\Phi(t,x)}{\varepsilon}\right)\,.$$

Let us first consider the boundary conditions in (1.1). Setting $u^{app,N_1,N_2}_\varepsilon|_{x_d=0} = \varepsilon\,v_\varepsilon(t,y)$ for simplicity, we compute

$$\begin{aligned} b(u^{app,N_1,N_2}_\varepsilon|_{x_d=0}) = &\sum_{n=1}^{N_1+N_2+1} \frac{\varepsilon^n}{n!} \sum_{\nu_1,\dots,\nu_n=0}^{N_1+N_2} {\rm d}^n b(0)\cdot(v_\varepsilon,\dots,v_\varepsilon) \\ &+ \frac{\varepsilon^{N_1+N_2+2}}{(N_1+N_2+1)!}\int_0^1 (1-s)^{N_1+N_2+1}\,{\rm d}^{N_1+N_2+2} b(\varepsilon\,s\,v_\varepsilon)\cdot(v_\varepsilon,\dots,v_\varepsilon)\,{\rm d}s\,. \end{aligned}$$

Collecting powers of ε and using the WKB cascade (4.3), we get

$$b(u_\varepsilon^{app,N_1,N_2}|_{x_d=0}) = \varepsilon^2 \int_0^1 (1-s)\, \mathrm{d}^2 b(\varepsilon\, s\, \mathcal{U}_0) \cdot (\mathcal{U}_0, \mathcal{U}_0) \left(t, y, 0, \frac{\varphi_0(t,y)}{\varepsilon}, \dots, \frac{\varphi_0(t,y)}{\varepsilon} \right) \mathrm{d}s \,,$$

if $N_1 = N_2 = 0$, and

$$\begin{aligned} b(u_\varepsilon^{app,N_1,N_2}|_{x_d=0}) &- \varepsilon^2\, G\left(t, y, \frac{\varphi_0(t,y)}{\varepsilon} \right) \\ &= \sum_{n=N_1+N_2+2}^{(N_1+N_2+1)^2} \varepsilon^n \sum_{\ell=1}^{N_1+N_2+1} \frac{1}{\ell !} \sum_{\substack{\nu_1+\cdots+\nu_\ell = n-\ell, \\ \nu_1,\dots,\nu_\ell \le N_1+N_2}} \mathrm{d}^\ell b(0) \cdot (\mathcal{U}_{\nu_1}, \dots, \mathcal{U}_{\nu_\ell}) \left(t, y, 0, \frac{\varphi_0(t,y)}{\varepsilon}, \dots, \frac{\varphi_0(t,y)}{\varepsilon} \right) \\ &\quad + \frac{\varepsilon^{N_1+N_2+2}}{(N_1+N_2+1)!} \int_0^1 (1-s)^{N_1+N_2+1}\, \mathrm{d}^{N_1+N_2+2} b(\varepsilon\, s\, v_\varepsilon) \cdot (v_\varepsilon, \dots, v_\varepsilon) \, \mathrm{d}s \,, \end{aligned}$$

if $N_1 + N_2 > 0$.

Each profile $\mathcal{U}_n$ belongs to $\mathcal{C}^\infty((-\infty,T]; H^{+\infty}(\mathbb{R}^d_+ \times (\mathbb{R}/(\Theta\, \mathbb{Z}))^N))$ and vanishes for $t<0$, hence it also belongs to $L^\infty((-\infty,T] \times (\mathbb{R}/(\Theta\, \mathbb{Z}))^N; L^2(\mathbb{R}^{d-1})) \cap L^\infty((-\infty,T] \times \mathbb{R}^{d-1} \times (\mathbb{R}/(\Theta\, \mathbb{Z}))^N)$ when restricted to the boundary $\{x_d=0\}$. We thus have uniform bounds with respect to ε of the type

$$\| \mathrm{d}^\ell b(0) \cdot (\mathcal{U}_{\nu_1}, \dots, \mathcal{U}_{\nu_\ell})(t,y,0,\varphi_0/\varepsilon, \dots, \varphi_0/\varepsilon) \|_{L^\infty((-\infty,T];L^2(\mathbb{R}^{d-1}))} \le C \,,$$

and similarly for the above integral remainders in Taylor's formula. When N_1+N_2 is positive, we thus get

$$\| b(u_\varepsilon^{app,N_1,N_2}|_{x_d=0}) - \varepsilon^2\, G(t,y,\varphi_0/\varepsilon) \|_{L^\infty((-\infty,T];L^2(\mathbb{R}^{d-1}))} \le C\, \varepsilon^{N_1+N_2+2} \,,$$

and we can derive the exact same $O(\varepsilon^{N_1+N_2+2})$ estimate for the L^∞ norm of the error at the boundary. When differentiating with respect to y, each partial derivative gives rise to a factor $1/\varepsilon$, which yields

$$\| b(u_\varepsilon^{app,N_1,N_2}|_{x_d=0}) - \varepsilon^2\, G(t,y,\varphi_0/\varepsilon) \|_{L^\infty((-\infty,T];H^{N_2}(\mathbb{R}^{d-1}))} \le C\, \varepsilon^{N_1+2} \,.$$

The case $N_1 = N_2 = 0$ is similar, except that the error is as large as the source term $\varepsilon^2\, G$, which is not so interesting from a practical point of view.

We leave to the interested reader the verification of the error estimate in the interior domain, which involves a little more algebra but no additional technical difficulty.

4.4 Extension to hyperbolic systems with constant multiplicity

The extension of the derivation of the leading amplitude equation (2.22) to hyperbolic systems with constant multiplicity is not entirely straightforward for two reasons, one related to the zero mean property of $\mathcal{U}_0$ and the other to the solvability of the interior profile equations.

When we analyzed the WKB cascade, the first point was to prove that the leading amplitude $\mathcal{U}_0$ necessarily has zero mean. This property does not extend obviously to the case of hyperbolic systems with constant multiplicity (similar issues arise in [CGW11]). Let us recall that when the system is hyperbolic with constant multiplicity, Lemma 2.1 and Lemma 2.2 hold. We then use the notation introduced in Paragraph 2.1 for the projectors P_m, Q_m and the partial inverses R_m. The only difference with the strictly hyperbolic framework is that we do not use the row vectors ℓ_m here. We now analyze the equations (2.6), (2.8) in this more general framework.

Equation (2.6)(a) shows that the leading profile $\mathcal{U}_0$ can be decomposed as

$$\mathcal{U}_0(t,x,\theta_1,\dots,\theta_M) = \underline{\mathcal{U}}_0(t,x) + \sum_{m=1}^{M} \mathcal{U}_0^m(t,x,\theta_m)\,, \quad P_m\,\mathcal{U}_0^m = \mathcal{U}_0^m\,,$$

where each $\mathcal{U}_0^m$ is Θ-periodic and has zero mean with respect to θ_m. The mean value $\underline{\mathcal{U}}$ satisfies the homogeneous boundary condition (2.11), and we are going to show that (2.10) still holds. Indeed, the mean value $\underline{\mathcal{U}}_0$ satisfies

$$L(\partial)\,\underline{\mathcal{U}}_0 + \underline{\mathcal{M}(\mathcal{U}_0,\mathcal{U}_0)} = 0\,,$$

which, in view of the decomposition of $\mathcal{U}_0$ and the definition (2.7) of $\mathcal{M}$, can be first simplified into

$$L(\partial)\,\underline{\mathcal{U}}_0 + \sum_{m=1}^{M} \underline{\mathcal{M}(\mathcal{U}_0^m,\mathcal{U}_0^m)} = 0\,.$$

We then compute

(4.16)

$$\mathcal{M}(\mathcal{U}_0^m,\mathcal{U}_0^m) = \partial_j\varphi_m\,(\mathrm{d}A_j(0)\cdot\mathcal{U}_0^m)\,\partial_{\theta_m}\mathcal{U}_0^m = \partial_j\varphi_m\,\mathrm{d}^2 f_j(0)\cdot(\mathcal{U}_0^m,\partial_{\theta_m}\mathcal{U}_0^m) = \frac{1}{2}\,\partial_j\varphi_m\,\partial_{\theta_m}\mathrm{d}^2 f_j(0)\cdot(\mathcal{U}_0^m,\mathcal{U}_0^m)\,,$$

where we have used Assumption 1.2 and the symmetry of $\mathrm{d}^2 f_j(0)$. Each $\mathcal{M}(\mathcal{U}_0^m,\mathcal{U}_0^m)$ therefore has mean zero and $\underline{\mathcal{U}}_0$ satisfies both (2.10) and (2.11) as in the strictly hyperbolic case. The result of Step 1 in Paragraph 2.2 thus extends to conservative hyperbolic systems with constant multiplicity.

We derive the interior equation for each $\mathcal{U}_0^m$ by retaining only the θ_m-oscillations in (2.6)(b) and applying the projector Q_m. Using [CG10, Lemma 3.3] and the absence of resonance, we get

$$(\partial_t + \mathbf{v}_m\cdot\nabla_x)\,Q_m\,\mathcal{U}_0^m + \frac{1}{2}\,\partial_j\varphi_m\,\partial_{\theta_m}Q_m\,\mathrm{d}^2 f_j(0)\cdot(\mathcal{U}_0^m,\mathcal{U}_0^m) = 0\,. \tag{4.17}$$

Let us also recall that the restriction of Q_m to Im P_m is injective, so that $Q_m\,\mathcal{U}_0^m$ uniquely determines $\mathcal{U}_0^m$ with $\mathcal{U}_0^m = P_m\,\mathcal{U}_0^m$. In spite of the symmetry of $d^2 f(0)$, (4.17) is not obviously a symmetric hyperbolic problem for the unknown $Q_m\mathcal{U}_0^m$, so its solvability is not immediately obvious. We encountered a similar difficulty in [CGW11], and we resolve it here in a similar way using the expression $\partial_j\varphi_m\,(\mathrm{d}A_j(0)\cdot\mathcal{U}_0^m)\partial_{\theta_m}\mathcal{U}_0^m$ for the nonlinear term and the following lemma.

Lemma 4.2. *Let $w\in\mathbb{R}^N$ and write $d\varphi_m = (\underline{\tau},\underline{\eta},\underline{\omega}_m) = (-\lambda_{k_m}(\underline{\xi}),\underline{\xi})$, where $\lambda_{k_m}(\underline{\xi}) = \lambda_{k_m}(u,\underline{\xi})|_{u=0}$. Then*

$$(Q_m\sum_{j=1}^{d}\underline{\xi}_j d_u A_j(0)\cdot w)P_m = (-d_u\lambda_{k_m}(0,\underline{\xi})\cdot w)Q_m P_m. \tag{4.18}$$

Proof. For u near 0 let $P_m(u)$ be the projector on Ker $\left(-\lambda_{k_m}(u,\underline{\xi})I + \sum_{j=1}^{d} A_j(u)\underline{\xi}_j\right)$ in the obvious decomposition of $\mathbb{C}^N$; thus, $P_m = P_m(0)$. Differentiate the equation

$$\left(-\lambda_{k_m}(u,\underline{\xi})I + \sum_{j=1}^{d} A_j(u)\underline{\xi}_j\right) P_m(u) = 0 \tag{4.19}$$

with respect to u in the direction w, evaluate at $u=0$, and apply Q_m on the left to obtain (4.18).

$\square$

Taking $w = \mathcal{U}_0^m$ in (4.18) and using $\mathcal{U}_0^m = P_m \mathcal{U}_0^m$, we see that (4.17) is a symmetric hyperbolic system (essentially scalar) for the unknown $Q_m \mathcal{U}_0^m$, so it can be solved with appropriate boundary and initial conditions just like the corresponding equations in the strictly hyperbolic case.

Equation (4.17) shows that all outgoing modes in $\mathcal{U}_0$ vanish, and there holds

$$\mathcal{U}_0(t,x,\theta_1,\dots,\theta_M) = \sum_{m\in\mathcal{I}} \mathcal{U}_0^m(t,x,\theta_m)\,, \quad P_m\,\mathcal{U}_0^m = \mathcal{U}_0^m\,. \tag{4.20}$$

In particular, there exists a scalar function a that is Θ-periodic with zero mean, such that

$$\forall\, m \in \mathcal{I}\,, \quad \mathcal{U}_0^m(t,y,0,\theta_0) = a(t,y,\theta_0)\, e_m, \text{ where } e_m = P_m e. \tag{4.21}$$

Our goal is to derive the amplitude equation that governs the evolution of a on the boundary.

At this stage, the analysis of Steps 3, 4, 5 in Paragraph 2.2 applies almost word for word, with obvious modifications in order to take into account the possibly many incoming and outgoing phases. Namely, we can show that the first corrector $\mathcal{U}_1$ has no outgoing mode. We can also determine the non-characteristic component $\mathcal{U}_1^{\mathrm{nc}}|_{x_d=0}$ and the non-polarized components $(I - P_m)\,\mathcal{U}_1^m|_{x_d=0}$ on the boundary in terms of the amplitude a. We end up with the exact same equation as (2.22), with a real constant v and a vector field X_{Lop} as in (2.23). The new expression of the bilinear operator Q_{per} reads (compare with (2.24)):

$$\begin{aligned} Q_{\mathrm{per}}[a,\widetilde{a}] &:= -\sum_{m\in\mathcal{O}} \sum_{\substack{m_1<m_2 \\ m_1,m_2\in\mathcal{I}}} \\ &\sum_{k\in\mathbb{Z}} \left(\sum_{\substack{k_{m_1}+k_{m_2}=k, \\ k_{m_1}\,k_{m_2}\neq 0}} \frac{k_{m_1}\,\underline{b}\,B\,A_d(0)^{-1}\,Q_m\,E_{m_1,m_2} + k_{m_2}\,\underline{b}\,B\,A_d(0)^{-1}\,Q_m\,E_{m_2,m_1}}{k_{m_1}\,(\underline{\omega}_{m_1}-\underline{\omega}_m) + k_{m_2}\,(\underline{\omega}_{m_2}-\underline{\omega}_m)}\, a_{k_{m_1}}\,\widetilde{a}_{k_{m_2}} \right) \mathrm{e}^{2\,i\,\pi\,k\,\theta_0/\Theta}\,, \end{aligned} \tag{4.22}$$

with vectors E_{m_1,m_2} as in (2.25).

The analysis of the WKB cascade (4.1), (4.3) proceeds as before, taking into account that incoming amplitudes $\mathcal{U}_j^m$ are propagated in the interior domain by vector-valued Burgers type equations, (4.17) and appropriate linearizations at $\mathcal{U}_0^m$, which can be solved using Lemma 4.2.

Part II
Pulses

5 Construction of approximate solutions

We follow the approach of Section 2 and first deal with strictly hyperbolic systems of three equations. We keep the notation of Paragraph 2.2, and make Assumption 2.3. Let us now derive the WKB cascade for pulse-like solutions to (1.1). The solution u_ε to (1.1) is assumed to have an asymptotic expansion of the form

$$u_\varepsilon \sim \varepsilon \sum_{k\geq 0} \varepsilon^k \, \mathcal{U}_k \left(t, x, \frac{\varphi_0(t,y)}{\varepsilon}, \frac{x_d}{\varepsilon} \right) . \tag{5.1}$$

We use the notation θ_0 as a placeholder for the fast variable φ_0/ε, and ξ_d for x_d/ε. Plugging the ansatz (5.1) in (1.1) and identifying powers of ε, we obtain the following first three relations for the $\mathcal{U}_k$'s (observe the slight differences with Paragraph 2.2, though we keep the same notation):

$$\begin{aligned}
&\text{(a)} \quad \mathcal{L}(\partial_{\theta_0}, \partial_{\xi_d}) \, \mathcal{U}_0 = 0 \, , \\
&\text{(b)} \quad \mathcal{L}(\partial_{\theta_0}, \partial_{\xi_d}) \, \mathcal{U}_1 + L(\partial) \, \mathcal{U}_0 + \mathcal{M}(\mathcal{U}_0, \mathcal{U}_0) = 0 \, , \\
&\text{(c)} \quad \mathcal{L}(\partial_{\theta_0}, \partial_{\xi_d}) \, \mathcal{U}_2 + L(\partial) \, \mathcal{U}_1 + \mathcal{M}(\mathcal{U}_0, \mathcal{U}_1) + \mathcal{M}(\mathcal{U}_1, \mathcal{U}_0) + \mathcal{N}_1(\mathcal{U}_0, \mathcal{U}_0) + \mathcal{N}_2(\mathcal{U}_0, \mathcal{U}_0, \mathcal{U}_0) = 0 \, ,
\end{aligned} \tag{5.2}$$

where the differential operators $\mathcal{L}, \mathcal{M}, \mathcal{N}_1, \mathcal{N}_2$ are now defined by:

$$\begin{aligned}
&\mathcal{L}(\partial_{\theta_0}, \partial_{\xi_d}) := A_d(0) \, (\partial_{\xi_d} + i \, \mathcal{A}(\underline{\tau}, \underline{\eta}) \, \partial_{\theta_0}) \, , \\
&\mathcal{M}(v, w) := \partial_j \varphi_0 \, (\mathrm{d}A_j(0) \cdot v) \, \partial_{\theta_0} w + (\mathrm{d}A_d(0) \cdot v) \, \partial_{\xi_d} w \, , \\
&\mathcal{N}_1(v, w) := (\mathrm{d}A_j(0) \cdot v) \, \partial_j w \, , \\
&\mathcal{N}_2(v, v, w) := \frac{1}{2} \, \partial_j \varphi_0 \, (\mathrm{d}^2 A_j(0) \cdot (v, v)) \, \partial_{\theta_0} w + \frac{1}{2} \, (\mathrm{d}^2 A_d(0) \cdot (v, v)) \, \partial_{\xi_d} w \, .
\end{aligned} \tag{5.3}$$

The equations (5.2) in the domain $(-\infty, T] \times \mathbb{R}^d_+ \times \mathbb{R}_{\theta_0} \times \mathbb{R}^+_{\xi_d}$ are supplemented with the boundary conditions obtained by plugging (5.1) in the boundary conditions of (1.1), which yields (recall $B = \mathrm{d}b(0)$):

$$\begin{aligned}
&\text{(a)} \quad B \, \mathcal{U}_0 = 0 \, , \\
&\text{(b)} \quad B \, \mathcal{U}_1 + \frac{1}{2} \, \mathrm{d}^2 b(0) \cdot (\mathcal{U}_0, \mathcal{U}_0) = G(t, y, \theta_0) \, ,
\end{aligned} \tag{5.4}$$

where functions on the left hand side of (5.4) are evaluated at $x_d = \xi_d = 0$. In order to get $u_\varepsilon|_{t<0} = 0$, as required in (1.1), we also look for profiles $\mathcal{U}_k$ that vanish for $t < 0$.

The construction of profiles in the wavetrain setting was accomplished by first assuming that solutions exist within the class of periodic functions of θ, and then using that assumption to actually construct periodic solutions. The construction of profiles in the pulse setting has the difficulty that it is not so clear at first in what function space(s) solutions should be sought. Construction of each successive pulse corrector involves an additional integration over a noncompact set. Thus, each corrector $\mathcal{U}_j$ is "worse" than the previous one $\mathcal{U}_{j-1}$, and successive correctors must be sought in successively larger spaces. In fact we will find that correctors beyond $\mathcal{U}_2$ are useless; they grow at least linearly in (θ_0, ξ_d), and are thus too large to be considered correctors.

We now define spaces $\mathcal{V}_F \subset \mathcal{V}_H \subset C^1_b$, where C^1_b is the space of C^1 functions $K(t,x,\theta_0,\xi_d)$, valued in $\mathbb{R}^3$, and bounded with their first-order derivatives. In the subsequent discussion we will assume $\mathcal{U}_0 \in \mathcal{V}_F$, $\mathcal{U}_1 \in \mathcal{V}_H$, and $\mathcal{U}_2 \in C^1_b$, and then construct profiles with those properties.

Definition 5.1. *a) Let $\mathcal{V}_F$ denote the space of functions $F(t,x,\theta_0,\xi_d) = \sum_{i=1}^3 F_i(t,x,\theta_0,\xi_d)A_d(0)r_i$, where each F_i is a finite sum of real-valued functions of the form*

$$f(t,x,\theta_0+\underline{\omega}_k\xi_d), \qquad g(t,x,\theta_0+\underline{\omega}_l\xi_d)h(t,x,\theta_0+\underline{\omega}_m\xi_d), \text{ and}$$
$$w(t,x,\theta_0+\underline{\omega}_p\xi_d)y(t,x,\theta_0+\underline{\omega}_q\xi_d)z(t,x,\theta_0+\underline{\omega}_r\xi_d), \tag{5.5}$$

where the indices $k,l,\dots,r$ lie in $\{1,2,3\}$.[8] *The functions $f(t,x,\theta)$, $g(t,x,\theta)$, etc., in (5.5) are C^1 and decay with their first-order partials at the rate $O(\langle\theta\rangle^{-2})$ uniformly with respect to (t,x). We refer to these functions as the "constituent functions" of $F \in \mathcal{V}_F$.*

b) Define a transversal interaction integral *to be a function $I^i_{l,m}(t,x,\theta_0,\xi_d)$ of the form*

$$I^i_{l,m}(t,x,\theta_0,\xi_d) = \int_{+\infty}^{\xi_d} \sigma(t,x,\theta_0+\underline{\omega}_i\xi_d+s(\underline{\omega}_l-\underline{\omega}_i))\tau(t,x,\theta_0+\underline{\omega}_i\xi_d+s(\underline{\omega}_m-\underline{\omega}_i))ds \tag{5.6}$$

where $\underline{\omega}_l$, $\underline{\omega}_m$, and $\underline{\omega}_i$ are mutually distinct.

c) Let $\mathcal{V}_H$ denote the space of functions $H(t,x,\theta_0,\xi_d) = \sum_{i=1}^3 H_i(t,x,\theta_0,\xi_d)A_d(0)r_i$ where each H_i is the sum of an element of $\mathcal{V}_F$ plus a finite sum of terms of the form

$$I^i_{l,m}(t,x,\theta_0,\xi_d) \text{ or } \alpha(t,x,\theta_0+\underline{\omega}_k\xi_d)J^n_{p,q}(t,x,\theta_0,\xi_d), \tag{5.7}$$

where $I^i_{l,m}$ and $J^n_{p,q}$ are transversal interaction integrals and the indices $i,l,\dots,q$ lie in $\{1,2,3\}$. The functions $\alpha(t,x,\theta)$, $\sigma(t,x,\theta)$, $\tau(t,x,\theta)$ are real-valued, C^1 and decay with their first-order partials at the rate $O(\langle\theta\rangle^{-3})$ uniformly with respect to (t,x).

d) For $H \in \mathcal{V}_H$ we can write $H = H_F + H_I$, where $H_F \in \mathcal{V}_F$ and $H_I \notin \mathcal{V}_F$ has components in the span of terms of the form (5.7). The "constituent functions" of H include those of H_F as well as the functions like α, σ, τ as in (5.6), (5.7) which constitute H_I.

Remark 5.2. *The same spaces of functions $\mathcal{V}_F$ and $\mathcal{V}_H$ are obtained if one begins by writing $F(t,x,\theta_0,\xi_d) = \sum_{i=1}^3 \tilde{F}_i(t,x,\theta_0,\xi_d)r_i$ and $H(t,x,\theta_0,\xi_d) = \sum_{i=1}^3 \tilde{H}_i(t,x,\theta_0,\xi_d)r_i$ and imposes the conditions in (a) (resp. (c)) on the $\tilde{F}_i$ (resp. $\tilde{H}_i$).*

5.1 Averaging and solution operators

To construct the profiles we must solve equations of the form $\mathcal{L}(\partial_{\theta_0},\partial_{\xi_d})\mathcal{U} = H$, where H lies in $\mathcal{V}_H$ and sometimes in $\mathcal{V}_F$. In this section we define averaging operators E_P and E_Q and a solution operator R_∞ (involving integration on a noncompact set) that provide a systematic way to study such equations. Henceforth we write $\mathcal{L} := \mathcal{L}(\partial_{\theta_0},\partial_{\xi_d})$.

The following simple lemma implies the existence of most of the limits and integrals that appear below.

Lemma 5.3. *Let $\sigma(t,x,\theta)$, $\tau(t,x,\theta)$ be continuous functions such that*

$$|\sigma(t,x,\theta)| \lesssim \langle\theta\rangle^{-N}, \quad |\tau(t,x,\theta)| \lesssim \langle\theta\rangle^{-N} \text{ for some } N \geq 2, \tag{5.8}$$

[8] Elements of $\mathcal{V}_F$ with terms involving no triple products were called functions of type F in [CW].

and suppose $\alpha(t,x,\theta_0,\xi_d,s)$ is continuous and bounded. Let i,l,m,q lie in $\{1,2,3\}$ and suppose i,l,m are mutually distinct. Then

$$\begin{aligned}
&(a)\ \int_{\xi_d}^{+\infty}|\sigma(t,x,\theta_0+\underline{\omega}_i\xi_d+r(\underline{\omega}_l-\underline{\omega}_i))|dr\lesssim 1,\\
&(b)\ \int_{\xi_d}^{+\infty}|\sigma(t,x,\theta_0+\underline{\omega}_i\xi_d+r(\underline{\omega}_l-\underline{\omega}_i))\tau(t,x,\theta_0+\underline{\omega}_i\xi_d+r(\underline{\omega}_m-\underline{\omega}_i))|dr\lesssim\langle\xi_d\rangle^{-N+1},\\
&(c)\ \text{and for } N\geq 3,\ \int_{\xi_d}^{+\infty}\int_s^{+\infty}|\sigma(t,x,\theta_0+\underline{\omega}_q\xi_d+s(\underline{\omega}_i-\underline{\omega}_q)+r(\underline{\omega}_l-\underline{\omega}_i))\cdot\\
&\qquad\tau(t,x,\theta_0+\underline{\omega}_q\xi_d+s(\underline{\omega}_i-\underline{\omega}_q)+r(\underline{\omega}_m-\underline{\omega}_i))|drds\lesssim\langle\xi_d\rangle^{-N+2}
\end{aligned}\tag{5.9}$$

Proof. Part (a) is immediate. To prove (b) set $\theta=\theta_0+\underline{\omega}_i\xi_d$ and let $\chi_1(\theta,r)$, $\chi_2(\theta,r)$ be nonnegative, smooth cutoffs summing to one and supported on the sets where $|\theta+r(\underline{\omega}_l-\underline{\omega}_i)|$ is respectively $\leq\frac{|(\underline{\omega}_l-\underline{\omega}_m)r|}{2}$, $\geq\frac{|(\underline{\omega}_l-\underline{\omega}_m)r|}{3}$. With the cutoffs inserted, in each case the integrand is $\lesssim\langle r\rangle^{-N}$. Part (c) is proved similarly using the same cutoffs, but now taking $\theta=\theta_0+\underline{\omega}_q\xi_d+s(\underline{\omega}_i-\underline{\omega}_q)$.

□

The definition of $\mathcal{V}_H$ and lemma 5.3 imply that the limits and integrals in the next definition are well-defined.

Definition 5.4 (E_P,E_Q,R_∞)**.** *For $H\in\mathcal{V}_H$ define averaging operators*

$$\begin{aligned}
E_QH(t,x,\theta_0,\xi_d)&=\sum_{j=1}^3\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T\ell_j\cdot H(x,\theta_0+\underline{\omega}_j(\xi_d-s),s)ds\right)A_d(0)r_j\\
E_PH(t,x,\theta_0,\xi_d)&=\sum_{j=1}^3\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T\ell_j\cdot A_d(0)H(x,\theta_0+\underline{\omega}_j(\xi_d-s),s)ds\right)r_j.
\end{aligned}\tag{5.10}$$

For $H\in\mathcal{V}_H$ such that $E_QH=0$ define the solution operator

$$R_\infty H(t,x,\theta_0,\xi_d)=\sum_{j=1}^3\left(\int_{+\infty}^{\xi_d}\ell_j\cdot H(t,x,\theta_0+\underline{\omega}_j(\xi_d-s),s)ds\right)r_j.\tag{5.11}$$

Remark 5.5. *(a) Suppose $F(x,\theta_0,\xi_d)=\sum_{i=1}^3F_i(x,\theta_0,\xi_d)A_d(0)r_i\in\mathcal{V}_F$, where each F_i has the form*

$$F_i(x,\theta_0,\xi_d)=\sum_{k=1}^3a_k^if_k^i(x,\theta_0+\omega_k\xi_d)+\sum_{l,m=1}^3b_{l,m}^ig_l^i(x,\theta_0+\omega_l\xi_d)h_m^i(x,\theta_0+\omega_m\xi_d),\tag{5.12}$$

Then $E_QF=\sum_{i=1}^3\tilde{F}_i(x,\theta_0,\xi_d)A_d(0)r_i$, where $\tilde{F}_i=a_i^if_i^i(x,\theta_0+\omega_i\xi_d)+b_{i,i}^ig_i^i(x,\theta_0+\omega_i\xi_d)h_i^i(x,\theta_0+\omega_i\xi_d)$. The obvious analogues hold when $F\in\mathcal{V}_F$ involves triple products, or when E_P is used in place of E_Q.

(b) Let $H\in\mathcal{V}_H$ and write $H=H_F+H_I$, where $H_F\in\mathcal{V}_F$ and $H_I\notin\mathcal{V}_F$ as in Definition 5.1(d). Then Lemma 5.3 implies $E_QH_I=0$ and $E_PH_I=0$.

Proposition 5.6. *Suppose $H\in\mathcal{V}_H$ and let $\mathcal{L}=\mathcal{L}(\partial_{\theta_0},\partial_{\xi_d})$.*

(a) $E_Q\mathcal{L}H=\mathcal{L}E_PH=0$.

(b) If $E_QH=0$ then $R_\infty H$ is bounded and $\mathcal{L}R_\infty H=H=(I-E_Q)H$.

(c) If $E_PH=0$ then $R_\infty\mathcal{L}H=H=(I-E_P)H$.

Proof. (a) $\mathcal{L}E_P H = 0$ follows directly from Remark 5.5 (a),(b). To show $E_Q\mathcal{L}H = 0$, write $H = \sum_{i=1}^3 H_i r_i$, and note that

$$\mathcal{L}(H_i r_i) = (\partial_{\xi_d} - \underline{\omega}_i \partial_{\theta_0}) H_i A_d(0) r_i. \tag{5.13}$$

Thus, the integrals $\int_0^T$ in the definition of $E_Q\mathcal{L}H$ can be evaluated and are uniformly bounded with respect to T.

(b) Boundedness of $R_\infty H$ follows from Lemma 5.3. Writing $H = \sum_{i=1}^3 H_i A_d(0) r_i$, a direct computation using $L(d\phi_i) r_i = 0$ shows $\mathcal{L}R_\infty H = H$.

(c) With $\beta = (\underline{\tau}, \underline{\eta})$ define $A(\beta)$ by $L(d\phi_k) = A(\beta) + A_d(0)\underline{\omega}_k$, and observe that

$$\ell_k A(\beta) r_j = \begin{cases} 0, \; j \neq k \\ -\underline{\omega}_k, \; j = k \end{cases}. \tag{5.14}$$

Writing $H = \sum_{j=1}^3 \tilde{H} r_i$ and using (5.14), direct computation yields

$$R_\infty \mathcal{L}H = \sum_{k=1}^3 \left(\int_{+\infty}^{\xi_d} (\partial_{\xi_d} - \underline{\omega}_k \partial_{\theta_0}) \tilde{H}(t, x, \theta_0 + \underline{\omega}_k(\xi_d - s), s) ds \right) r_k = \sum_k \tilde{H}_k r_k, \tag{5.15}$$

since $\lim_{M\to+\infty} \tilde{H}_k(t, x, \theta_0 + \underline{\omega}_k(\xi_d - M), M) = 0$ when $E_P H = 0$.

□

The next proposition, which extends Proposition 1.22 of [CW], is used repeatedly in constructing $\mathcal{U}_0$, $\mathcal{U}_1$, and $\mathcal{U}_2$.

Proposition 5.7. *Suppose $H \in \mathcal{V}_H$.*

(a) The equation $\mathcal{L}\mathcal{U} = H$ has a bounded C^1 solution if and only if $E_Q H = 0$.[9]

(b) When $E_Q H = 0$, every bounded C^1 solution has the form

$$\mathcal{U} = \sum_{i=1}^3 \tau_i(t, x, \theta_0 + \underline{\omega}_i \xi_d) r_i + R_\infty H. \tag{5.16}$$

(c) When H as in part (b) belongs to $\mathcal{V}_F$, we have $E_P\mathcal{U} = \sum_{i=1}^3 \tau_i(t, x, \theta_0 + \underline{\omega}_i \xi_d) r_i$ and $(I - E_P)\mathcal{U} = R_\infty H$.

Proof. (a) The direction $\Leftarrow$ is given by Proposition 5.6(b). ($\Rightarrow$) Suppose there is a bounded C^1 solution to $\mathcal{L}U = H$, and write $H = H_F + H_I$ as in Remark 5.5(b). Since $E_Q H_I = 0$, there is a bounded C^1 solution to $\mathcal{L}U = H_I$, so we conclude there is a bounded C^1 solution to $\mathcal{L}U = H_F$, and similarly to $\mathcal{L}U = E_Q H_F$. From the explicit form of $E_Q H_F$ given in Remark 5.5(a), we see that there is no bounded C^1 solution of $\mathcal{L}U = E_Q H_F$ if $E_Q H_F \neq 0$. Thus, $E_Q H_F = 0$ and hence $E_Q H = 0$.

(b) By Proposition 5.6(b) $R_\infty H$ is a bounded C^1 solution of $\mathcal{L}U = H$; moreover, the general C^1 solution of $\mathcal{L}U = 0$ has the form $\sum_{i=1}^3 \tau_i(t, x, \theta_0 + \underline{\omega}_i \xi_d) r_i$.

(c) Lemma 5.3 implies $E_P R_\infty H = 0$ when $H \in \mathcal{V}_F$ and $E_Q H = 0$.

□

[9]The proof shows that $\mathcal{L}U = H$ has a C^1 solution sublinear with respect to ξ_d if and only if $E_Q H = 0$.

5.2 Profile construction and proof of Theorem 1.11.

For $\Omega_T := (-\infty, T] \times \mathbb{R}^d_x \times \mathbb{R}_\theta$ we define the weighted Sobolev spaces:

$$\Gamma^k(\Omega_T) := \left\{ u \in L^2(\Omega_T) \,:\, \theta^\alpha\, \partial^\beta_{t,x,\theta} u \in L^2(\Omega_T) \quad \text{if } \alpha + |\beta| \le k \right\}.$$

This is a Hilbert space for the norm

$$\|u\|^2_{\Gamma^k(\Omega_T)} := \sum_{\alpha+|\beta|\le k} \|\theta^\alpha\, \partial^\beta_{t,x,\theta} u\|^2_{L^2(\Omega_T)} .$$

For $L \in \mathbb{N}$ when $k > \frac{d+2}{2} + L + 1$, elements of $\Gamma^k(\Omega_T)$ are C^1 and decay with their first-order partials at the rate $\langle\theta\rangle^{-L}$ uniformly with respect to $(t.x)$.

Definition 5.8. *(a) Suppose $k > \frac{d+2}{2} + 3$. Let $\mathcal{V}^k_F \subset \mathcal{V}_F$ be the subspace consisting of elements whose constituent functions lie in $\Gamma^k(\Omega_T)$.*

(b) Suppose $k > \frac{d+2}{2} + 4$. Let $\mathcal{V}^k_H \subset \mathcal{V}_H$ be the subspace consisting of elements whose constituent functions lie in $\Gamma^k(\Omega_T)$.

For $K_0 > 8 + \frac{d+2}{2}$ we assume now that the equations (5.2), (5.4) have solutions $\mathcal{U}_0 \in \mathcal{V}_F$, $\mathcal{U}_1 \in \mathcal{V}_H$, $\mathcal{U}_2 \in C^1_b$ such that

$$(5.17)\qquad \begin{aligned} &(a)\ \mathcal{U}_0 \in \mathcal{V}^{K_0}_F,\ \mathcal{U}_0 = E_P\mathcal{U}_0 = \sum_{j=1}^3 \sigma_j(t,x,\theta_0 + \underline{\omega}_j \xi_d) r_j, \text{ where } \underline{\sigma}_j = \int_{-\infty}^{\infty} \sigma_j d\theta = 0, \\ &(b)\ E_P\mathcal{U}_1 \in \mathcal{V}^{K_0-3}_F \text{ and } (I - E_P)\mathcal{U}_1 \in \mathcal{V}^{K_0-2}_H, \end{aligned}$$

and then construct solutions with those properties.

Step 1: the leading profile $\mathcal{U}_0$ has no outgoing component.

This step justifies one of the causality arguments used in [MR83]. Equation 5.2(a) and Proposition 5.7(b) imply that the expression of $\mathcal{U}_0$ reduces to:

$$(5.18)\qquad \mathcal{U}_0(t,x,\theta_0,\xi_d) = \sum_{m=1}^3 \sigma_m(t,x,\theta_0 + \underline{\omega}_m\, \xi_d)\, r_m$$

for functions σ_m to be determined. The last variable of σ_m is denoted θ_m in what follows.

Since $\mathcal{U}_1$ is a bounded solution to $\mathcal{L}\mathcal{U}_1 = \mathcal{F}_0$, where $\mathcal{F}_0 = -[L(\partial)\mathcal{U}_0 + \mathcal{M}(\mathcal{U}_0,\mathcal{U}_0)] \in \mathcal{V}^{K_0-1}_F$, Proposition 5.7(a) implies $E_Q\mathcal{F}_0 = 0$; that is,

$$(5.19)\qquad \partial_t\sigma_m + \mathbf{v}_m \cdot \nabla_x \sigma_m + c_m\, \sigma_m\, \partial_{\theta_m} \sigma_m = 0\,, \quad m = 1,2,3\,, \quad c_m := \frac{\partial_j\varphi_m\, \ell_m\, (\mathrm{d}A_j(0)\cdot r_m)\, r_m}{\ell_m\, r_m} .$$

Since φ_2 is outgoing, this implies $\sigma_2 \equiv 0$, and the boundary condition (5.4)(a) gives, as in Paragraph 2.2, the existence of a scalar function a such that

$$(5.20)\qquad \sigma_1(t,y,0,\theta_0)\, r_1 = a(t,y,\theta_0)\, e_1\,, \quad \sigma_3(t,y,0,\theta_0)\, r_3 = a(t,y,\theta_0)\, e_3\,.$$

Step 2: showing $(I - E_P)\mathcal{U}_1 \in \mathcal{V}^{K_0-2}_H$.

At this stage, we know that the leading profile $\mathcal{U}_0$ reads

$$\mathcal{U}_0(t,x,\theta_0,\xi_d) = \sigma_1(t,x,\theta_0+\underline{\omega}_1\,\xi_d)\,r_1 + \sigma_3(t,x,\theta_0+\underline{\omega}_3\,\xi_d)\,r_3\,, \tag{5.21}$$

where σ_1, σ_3 satisfy (5.19) and their traces satisfy (5.20). We thus compute

$$\mathcal{F}_0 = -L(\partial)\,(\sigma_1\,r_1+\sigma_3\,r_3) - \sigma_1\,\partial_{\theta_1}\sigma_1\,R_{1,1} - \sigma_3\,\partial_{\theta_3}\sigma_3\,R_{3,3} - \sigma_3\,\partial_{\theta_1}\sigma_1\,R_{1,3} - \sigma_1\,\partial_{\theta_3}\sigma_3\,R_{3,1}\,, \tag{5.22}$$

with

$$\forall\, m_1,m_2=1,3\,,\quad R_{m_1,m_2} := \partial_j\varphi_{m_1}\,(\mathrm{d}A_j(0)\cdot r_{m_2})\,r_{m_1}\,,$$

and the functions $(\sigma_1,\partial_{\theta_1}\sigma_1)$, resp. $(\sigma_3,\partial_{\theta_3}\sigma_3)$, in (5.22) are evaluated at $(t,x,\theta_0+\underline{\omega}_1\,\xi_d)$, resp. $(t,x,\theta_0+\underline{\omega}_3\,\xi_d)$. By Proposition 5.7(b) we have

$$\begin{aligned}
&\mathcal{U}_1 = \sum_{i=1}^{3}\tau_i(t,x,\theta_0+\underline{\omega}_i\xi_d)r_i + R_\infty\mathcal{F}_0,\\
&(I-E_P)\mathcal{U}_1 = R_\infty\mathcal{F}_0 = \sum_{i=1}^{3}\left(\int_{+\infty}^{\xi_d}F_i(t,x,\theta_0+\underline{\omega}_i(\xi_d-s),s)ds\right)r_i.
\end{aligned} \tag{5.23}$$

Since $E_Q\mathcal{F}_0=0$, the integrand $F_i(t,x,\theta_0+\underline{\omega}_i(\xi_d-s),s)=$

$$\begin{aligned}
&\sum_{k\neq i}V^i_k\sigma_k(t,x,\theta_0+\omega_i\xi_d+s(\omega_k-\omega_i))+\\
&\sum_{k\neq i}c^i_k\sigma_k(t,x,\theta_0+\omega_i\xi_d+s(\omega_k-\omega_i))\partial_{\theta_k}\sigma_k(t,x,\theta_0+\omega_i\xi_d+s(\omega_k-\omega_i))+\\
&\sum_{m\neq i}d^i_{i,m}\sigma_i(t,x,\theta_0+\omega_i\xi_d)\partial_{\theta_m}\sigma_m(t,x,\theta_0+\omega_i\xi_d+s(\omega_m-\omega_i))+\\
&\sum_{l\neq i}d^i_{l,i}\sigma_l(t,x,\theta_0+\omega_i\xi_d+s(\omega_l-\omega_i))\partial_{\theta_i}\sigma_i(t,x,\theta_0+\omega_i\xi_d)+\\
&\sum_{l\neq m,l\neq i,m\neq i}d^i_{l,m}\sigma_l(t,x,\theta_0+\omega_i\xi_d+s(\omega_l-\omega_i))\partial_{\theta_m}\sigma_m(t,x,\theta_0+\omega_i\xi_d+s(\omega_m-\omega_i)),
\end{aligned} \tag{5.24}$$

where the c^i_k, $d^i_{l,m}$ are real constants, and V^i_k is the tangential vector field given by $V^i_k\sigma_k=\ell_iL(\partial)(\sigma_kr_k)$. After integrating $\int_{+\infty}^{\xi_d}(\cdot)ds$ the second and third lines of (5.24), constituent functions in $\mathcal{V}^{K_0}_F$ are obtained. The first and fourth lines can be integrated using the moment zero property of the σ_j to yield constituent functions in $\mathcal{V}^{K_0-2}_F$ and $\mathcal{V}^{K_0-1}_F$ respectively.[10] The integral of the fifth line is a linear combination of transversal interaction integrals with constituent functions in $\mathcal{V}^{K_0-1}_F$.

Step 3: equations for $E_P\mathcal{U}_1$.

Since $\mathcal{U}_2\in C^1_b$ is a solution of $\mathcal{L}\mathcal{U}_2=\mathcal{F}_1$, where

$$\mathcal{F}_1 := -[L(\partial)\,\mathcal{U}_1+\mathcal{M}(\mathcal{U}_0,\mathcal{U}_1)+\mathcal{M}(\mathcal{U}_1,\mathcal{U}_0)+\mathcal{N}_1(\mathcal{U}_0,\mathcal{U}_0)+\mathcal{N}_2(\mathcal{U}_0,\mathcal{U}_0,\mathcal{U}_0)]\in\mathcal{V}^{K_0-4}_H, \tag{5.25}$$

[10] Without this moment zero property, these integrals and thus $\mathcal{U}_1$ would be no better than bounded; consequently, $\mathcal{U}_2$ would be unbounded.

Proposition 5.7(a) implies $E_Q\mathcal{F}_1 = 0$. We rewrite this and include the boundary condition on $\mathcal{U}_1$ to obtain

$$
\begin{aligned}
&(a)\ E_Q[L(\partial)E_P\mathcal{U}_1 + \mathcal{M}(\mathcal{U}_0, E_P\mathcal{U}_1) + \mathcal{M}(E_P\mathcal{U}_1, \mathcal{U}_0)] = \\
&\quad - E_Q[L(\partial)(I - E_P)\mathcal{U}_1 + \mathcal{M}(\mathcal{U}_0, (I - E_P)\mathcal{U}_1) + \mathcal{M}((I - E_P)\mathcal{U}_1, \mathcal{U}_0) + \mathcal{N}_1 + \mathcal{N}_2] \\
&(b)\ BE_P\mathcal{U}_1 = G - \frac{1}{2}d^2b(0)(\mathcal{U}_0, \mathcal{U}_0) - B(I - E_P)\mathcal{U}_1 \text{ on } x_d = \xi_d = 0.
\end{aligned}
\tag{5.26}
$$

Letting $V := (I - E_P)\mathcal{U}_1$ and decomposing $V = V_F + V_I$ as in Definition 5.1(d), we can use Lemma 5.3(c) to see that

$$
E_Q[L(\partial)V_I + \mathcal{M}(\mathcal{U}_0, V_I) + \mathcal{M}(V_I, \mathcal{U}_0) = 0. \tag{5.27}
$$

Thus, (5.26)(a) simplifies to

$$
\begin{aligned}
&E_Q[L(\partial)E_P\mathcal{U}_1 + \mathcal{M}(\mathcal{U}_0, E_P\mathcal{U}_1) + \mathcal{M}(E_P\mathcal{U}_1, \mathcal{U}_0)] = \\
&\qquad - E_Q[L(\partial)V_F + \mathcal{M}(\mathcal{U}_0, V_F) + \mathcal{M}(V_F, \mathcal{U}_0) + \mathcal{N}_1 + \mathcal{N}_2] \in \mathcal{V}_F^{K_0-3}.
\end{aligned}
\tag{5.28}
$$

The components of $E_P\mathcal{U}_1$ will be determined from equations (5.28) and (5.26)(b).

Step 4: determining the component of $E_P\mathcal{U}_1$ on r_2.

We now show $\tau_2 = 0$, thereby justifying another causality argument in [MR83]. Since $\mathcal{U}_0$ is purely incoming, using Remark 5.5(a) we see that the second component of the right side of (5.28) is zero. Similarly, the second component of $E_Q[\mathcal{M}(\mathcal{U}_0, E_P\mathcal{U}_1) + \mathcal{M}(E_P\mathcal{U}_1, \mathcal{U}_0)]$ is zero. This forces τ_2 to satisfy the homogeneous transport equation

$$
(\partial_t + \mathbf{v}_2 \cdot \nabla_x)\, \tau_2 = 0\,,
$$

Since φ_2 is an outgoing phase, τ_2 is identically zero.

Remark 5.9. *The justification of $\tau_2 = 0$ relies on the assumption that (5.2)(c) admits a bounded solution $\mathcal{U}_2$. Boundedness of $\mathcal{U}_2$ makes the expression $\varepsilon^3\,\mathcal{U}_2$ meaningful as a corrector to the approximate solution $\varepsilon\,\mathcal{U}_0 + \varepsilon^2\,\mathcal{U}_1$. However, we shall see in Appendix B that assuming boundedness of $\mathcal{U}_2$ has a major consequence on the leading order profile $\mathcal{U}_0$. In particular, the governing equation (5.30) below for the evolution of a on the boundary will not coincide with the equation obtained by considering (2.19) in Part I in the regime $\Theta \to +\infty$.*

Step 5: the Mach stem equation for $a(t, y, \theta_0)$.

From the previous step of the analysis, the trace of the first corrector $\mathcal{U}_1$ satisfies

$$
\mathcal{U}_1(t, y, 0, \theta_0, 0) = \star\, r_1 - \int_0^{+\infty} \ell_2\, \mathcal{F}_0(t, y, 0, \theta_0 - \underline{\omega}_2\, X, X)\, \mathrm{d}X\, r_2 + \star\, r_3\,,
$$

where $\star$ denotes a coefficient whose expression is not useful for what follows, and $\mathcal{F}_0$ is given by (5.22). Plugging the latter expression in (5.4)(b) and applying the row vector $\underline{b}$, we get

$$
-\,\underline{b}\, B\, r_2 \int_0^{+\infty} \ell_2\, \mathcal{F}_0(t, y, 0, \theta_0 - \underline{\omega}_2\, X, X)\, \mathrm{d}X + \frac{1}{2}\, \underline{b}\, \mathrm{d}^2 b(0) \cdot (e, e)\, a^2 = \underline{b}\, G\,, \tag{5.29}
$$

which is the solvability condition for $E_P \mathcal{U}_1|_{x_d=\xi_d=0}$ in (5.26). It remains to differentiate (5.29) with respect to θ_0 and to identify the first term on the left hand side of (5.29). More precisely, we compute

$$\begin{aligned}\ell_2\,\mathcal{F}_0(t,y,0,\theta_0,\xi_d) = &-\ell_2\,L_{\tan}(\partial)\,(a(t,y,\theta_0+\underline{\omega}_1\,\xi_d)\,e_1 + a(t,y,\theta_0+\underline{\omega}_3\,\xi_d)\,e_3)\\ &-\frac{1}{2}\,\ell_2\,E_{1,1}\,\partial_{\theta_0}(a^2)(t,y,\theta_0+\underline{\omega}_1\,\xi_d) - \frac{1}{2}\,\ell_2\,E_{3,3}\,\partial_{\theta_0}(a^2)(t,y,\theta_0+\underline{\omega}_3\,\xi_d)\\ &-\ell_2\,E_{1,3}\,(\partial_{\theta_0}a)(t,y,\theta_0+\underline{\omega}_1\,\xi_d)\,a(t,y,\theta_0+\underline{\omega}_3\,\xi_d)\\ &-\ell_2\,E_{3,1}\,a(t,y,\theta_0+\underline{\omega}_1\,\xi_d)\,(\partial_{\theta_0}a)(t,y,\theta_0+\underline{\omega}_3\,\xi_d)\,,\end{aligned}$$

with $E_{1,1}, E_{3,3}, E_{1,3}, E_{3,1}$ as in (2.25), and

$$L_{\tan}(\partial) := \partial_t + \sum_{j=1}^{d-1} A_j(0)\,\partial_j\,.$$

Using the expression of the matrices R_1, R_3 in Paragraph 2.1, we thus find that (5.29) reduces to

$$\upsilon\,\partial_{\theta_0}(a^2) - X_{\rm Lop}a + \partial_{\theta_0}Q_{\rm pul}[a,a] = \underline{b}\,\partial_{\theta_0}G\,, \tag{5.30}$$

with υ as in (2.16), $X_{\rm Lop}$ as in (2.17), and[11]

$$\begin{aligned}Q_{\rm pul}[a,\widetilde{a}](\theta_0) := (\underline{b}\,B\,r_2)\,\ell_2\,E_{1,3}\int_0^{+\infty}\partial_{\theta_0}a(\theta_0+(\underline{\omega}_1-\underline{\omega}_2)\,X)\,\widetilde{a}(\theta_0+(\underline{\omega}_3-\underline{\omega}_2)\,X)\,{\rm d}X\\ +(\underline{b}\,B\,r_2)\,\ell_2\,E_{3,1}\int_0^{+\infty}a(\theta_0+(\underline{\omega}_1-\underline{\omega}_2)\,X)\,\partial_{\theta_0}\widetilde{a}(\theta_0+(\underline{\omega}_3-\underline{\omega}_2)\,X)\,{\rm d}X\,.\end{aligned} \tag{5.31}$$

Step 6: completing the construction of $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{U}_2$.

It is proved in Corollary 6.5 of section 6 that, with $K_0, K_1 \in \mathbb{N}$ and G as in Theorem 1.11, there exists $T > 0$ and a unique solution

$$a \in \cap_{\ell=0}^{K_0}\,\mathcal{C}^\ell((-\infty,T];\Gamma^{K_1-1-\ell}(\mathbb{R}^d_{y,\theta}))\,,$$

to (5.30), (5.31). As in (5.20) this determines the boundary data of σ_j, $j=1,3$. Corollary 6.6 of section 6 yields $\sigma_j \in \Gamma^{K_0}(\Omega_T)$ satisfying 5.19. Moreover, a and thus σ_j are shown there to have moment zero. That completes the construction of $\mathcal{U}_0$ with the properties in (5.17).

Assuming these results for now, we complete the construction of $\mathcal{U}_1$ and $\mathcal{U}_2$. With $\mathcal{U}_0$ determined, $(I-E_P)\mathcal{U}_1 \in \mathcal{V}_H^{K_0-2}$ is now constructed as in Step 2. To determine $E_P\mathcal{U}_1$ we return to (5.28) and (5.26)(b), noting that the right side of (5.28) is now determined. Writing $(I-E_P)\mathcal{U}_1 = V_F + V_I$ as before, we have $V_F|_{x_d=\xi_d=0} \in \Gamma^{K_0-3}$,[12] and the same holds for the trace of V_I as a consequence of Corollary 6.3. The right side of (5.26)(b) satisfies the required solvability condition, so that equation uniquely determines $E_P\mathcal{U}_1|_{x_d=\xi_d=0} \in \Gamma^{K_0-3}$, taking its value in $\check{\mathbb{E}}^s(\underline{\tau},\underline{\eta})$; recall (2.4). Equations 5.28 and (5.26)(b) determine decoupled transport equations for the components τ_1, τ_3 of $E_P\mathcal{U}_1$, and we obtain $\tau_j \in \Gamma^{K_0-3}$, and hence $E_P\mathcal{U}_1 \in \mathcal{V}_F^{K_0-3}$.

The profile $\mathcal{U}_2$ satisfies $\mathcal{L}\mathcal{U}_2 = \mathcal{F}_1$, where $\mathcal{F}_1$ as in (5.25) satisfies $E_Q\mathcal{F}_1 = 0$. Thus, Proposition 5.7 yields a solution $\mathcal{U}_2 = R_\infty\mathcal{F}_1 \in C^1_b$.

Apart from the results proved in section 6 that were used in this step, this completes the proof of Theorem 1.11 in the 3×3 strictly hyperbolic case. The profile $\mathcal{U}_0$ satisfies (5.2)(a), (5.4)(a), and the correctors $\mathcal{U}_1$, $\mathcal{U}_2$ satisfy (5.2)(b), (c) and (5.4)(b).

[11]The variables (t,y) enter as parameters in the definition of $Q_{\rm pul}$ so we omit them.

[12]We use self-explanatory notation here.

6 Analysis of the amplitude equation

6.1 Preliminary reductions

Our goal in this section is to prove a well-posedness result for the “Mach stems equation” (5.30). Once again, we focus on the case of 3×3 hyperbolic systems, and leave the minor modifications for the extension to $N \times N$ systems to the interested reader. Up to dividing by nonzero constants, and using the shorter notation θ instead of θ_0, Equation (5.30) takes the form

$$\partial_t a + \mathbf{w} \cdot \nabla_y a + c\, a\, \partial_\theta a + \partial_\theta Q_{\mathrm{pul}}[a,a] = g \,, \tag{6.1}$$

with $\mathbf{w} \in \mathbb{R}^{d-1}$, $c \in \mathbb{R}$, and the quadratic operator Q_{pul} is defined by

$$Q_{\mathrm{pul}}[a,\widetilde{a}](\theta) := \mu_1 \int_0^{+\infty} \partial_\theta a(\theta + (\underline{\omega}_1 - \underline{\omega}_2)\, X)\, \widetilde{a}(\theta + (\underline{\omega}_3 - \underline{\omega}_2)\, X)\, \mathrm{d}X \\ + \mu_3 \int_0^{+\infty} a(\theta + (\underline{\omega}_1 - \underline{\omega}_2)\, X)\, \partial_\theta \widetilde{a}(\theta + (\underline{\omega}_3 - \underline{\omega}_2)\, X)\, \mathrm{d}X \,,$$

where $\mu_1, \mu_3 \in \mathbb{R}$ and the $\underline{\omega}_m$'s are pairwise distinct. The latter operator only acts on the θ-variable, and (t,y) only enter as parameters, which we do not write for simplicity.

We first reduce the expression of Q_{pul} by recalling that $\underline{\omega}_1, \underline{\omega}_3$ are the two incoming modes while $\underline{\omega}_2$ is the outgoing mode. There is no loss of generality in assuming $\underline{\omega}_3 > \underline{\omega}_1$. Then we define the two nonzero parameters

$$\delta_1 := \frac{\underline{\omega}_1 - \underline{\omega}_2}{\underline{\omega}_3 - \underline{\omega}_1} \,, \quad \delta_3 := \frac{\underline{\omega}_3 - \underline{\omega}_2}{\underline{\omega}_3 - \underline{\omega}_1} \,,$$

that satisfy $\delta_3 = 1 + \delta_1$. Changing variables in the expression of Q_{pul} and redefining the constants $\mu_{1,3}$, we obtain

$$Q_{\mathrm{pul}}[a,\widetilde{a}](\theta) = \mu_1 \int_0^{+\infty} \partial_\theta a(\theta + \delta_1\, X)\, \widetilde{a}(\theta + \delta_3\, X)\, \mathrm{d}X + \mu_3 \int_0^{+\infty} a(\theta + \delta_1\, X)\, \partial_\theta \widetilde{a}(\theta + \delta_3\, X)\, \mathrm{d}X \,.$$

For later use, we define the following bilinear operator $\mathbb{F}_{\mathrm{pul}}$ acting on functions that depend on the variable $\theta \in \mathbb{R}$ (whenever the formula below makes sense):

$$\mathbb{F}_{\mathrm{pul}}(u,v)(\theta) := \int_0^{+\infty} u(\theta + \delta_1\, X)\, v(\theta + \delta_3\, X)\, \mathrm{d}X \,. \tag{6.2}$$

The operator $\mathbb{F}_{\mathrm{pul}}$ satisfies the properties:

$$\text{(Differentiation)} \quad \partial_\theta(\mathbb{F}_{\mathrm{pul}}(u,v)) = \mathbb{F}_{\mathrm{pul}}(\partial_\theta u, v) + \mathbb{F}_{\mathrm{pul}}(u, \partial_\theta v) \,, \tag{6.3}$$

$$\text{(Integration by parts)} \quad \mathbb{F}_{\mathrm{pul}}(u, \partial_\theta v) = -\frac{1}{\delta_3}\, u\, v - \frac{\delta_1}{\delta_3}\, \mathbb{F}_{\mathrm{pul}}(\partial_\theta u, v) \,. \tag{6.4}$$

Using the properties (6.3), (6.4), we can rewrite Equation (6.1) as

$$\partial_t a + \mathbf{w} \cdot \nabla_y a + c\, a\, \partial_\theta a + \mu\, \mathbb{F}_{\mathrm{pul}}(\partial_\theta a, \partial_\theta a) = g \,, \tag{6.5}$$

with suitable constants that are denoted c and μ for simplicity and whose exact expression is useless. Our goal is to solve Equation (6.5) by a standard fixed point argument. The main ingredient in the proof is to show that the nonlinear term $\mathbb{F}_{\mathrm{pul}}(\partial_\theta a, \partial_\theta a)$ acts as a *semilinear* term in a suitable scale of Sobolev regularity.

6.2 Boundedness of the bilinear operator $\mathbb{F}_{\text{pul}}$

The operator $\mathbb{F}_{\text{pul}}$ is not symmetric but changing the roles of δ_1 and δ_3, the roles of the first and second argument of $\mathbb{F}_{\text{pul}}$ in the estimates below can be exchanged. This will be used in several places.

Let us first recall the definition of weighted Sobolev spaces:

$$\Gamma^k(\mathbb{R}^d) := \Big\{ u \in L^2(\mathbb{R}^{d-1}_y \times \mathbb{R}_\theta) \, : \, \theta^\alpha \, \partial_{y,\theta}^\beta u \in L^2(\mathbb{R}^d) \quad \text{if } \alpha + |\beta| \le k \Big\} \, .$$

This is a Hilbert space for the norm

$$\|u\|^2_{\Gamma^k(\mathbb{R}^d)} := \sum_{\alpha+|\beta|\le k} \|\theta^\alpha \, \partial_{y,\theta}^\beta u\|^2_{L^2(\mathbb{R}^d)} \, .$$

Following the same integration by parts arguments as in [CW13, Proposition 3.3], there holds

Lemma 6.1. *For all integer k, the space $\Gamma^k(\mathbb{R}^d)$ coincides with*

$$\Big\{ u \in H^k(\mathbb{R}^{d-1}_y \times \mathbb{R}_\theta) \, : \, \theta^k \, u \in L^2(\mathbb{R}^d) \Big\} \, ,$$

and the norm of $\Gamma^k(\mathbb{R}^d)$ is equivalent to the norm

$$\|\theta^k \, u\|_{L^2(\mathbb{R}^d)} + \|u\|_{H^k(\mathbb{R}^d)} \, .$$

Our main boundedness result for the operator $\mathbb{F}_{\text{pul}}$ reads as follows.

Proposition 6.2. *Let k_0 denote the smallest integer satisfying $k_0 > (d+1)/2$. Then for all $k \ge 2\,k_0+1$, there exists a constant C_k satisfying*

$$\forall \, u, v \in \Gamma^k(\mathbb{R}^d) \, , \quad \|\mathbb{F}_{\text{pul}}(\partial_\theta u, \partial_\theta v)\|_{\Gamma^k(\mathbb{R}^d)} \le C_k \, \|u\|_{\Gamma^k(\mathbb{R}^d)} \, \|v\|_{\Gamma^k(\mathbb{R}^d)} \, . \tag{6.6}$$

The estimate (6.6) of Proposition 6.2 is not tame, but it will be sufficient for our purpose since we shall only construct finitely many terms in the WKB expansion of the solution u_ε to (1.1) (opposite to what we did in Part I where we constructed approximate solutions of arbitrarily high order).

Proof. Let us first observe that when u, v belong to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, $\mathbb{F}_{\text{pul}}(u,v)$ also belongs to $\mathcal{S}(\mathbb{R}^d)$. By a density/continuity argument, we are thus reduced to proving the estimate (6.6) for $u, v \in \mathcal{S}(\mathbb{R}^d)$. The decay and regularity of u, v will justify all the manipulations below.

1) We start with the basic L^2 estimate of the function $\mathbb{F}_{\text{pul}}(u,v)$. Using Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\Big| \int_0^{+\infty} u(y, \theta + \delta_1 \, X) \, v(y, \theta + \delta_3 \, X) \, \mathrm{d}X \Big|^2 \\
&\qquad \le \int_0^{+\infty} |u(y, \theta + \delta_1 \, X)| \, \mathrm{d}X \int_0^{+\infty} |u(y, \theta + \delta_1 \, X)| \, |v(y, \theta + \delta_3 \, X)|^2 \, \mathrm{d}X \\
&\qquad \le \frac{1}{|\delta_1|} \int_{\mathbb{R}} |u(y, \theta')| \, \mathrm{d}\theta' \int_{\mathbb{R}} |u(y, \theta + \delta_1 \, X)| \, |v(y, \theta + \delta_3 \, X)|^2 \, \mathrm{d}X \, .
\end{aligned}$$

Integrating with respect to (y,θ), and changing variables (use $\delta_3-\delta_1=1$), we get

$$\begin{aligned}\|\mathbb{F}_{\rm pul}(u,v)\|_{L^2(\mathbb{R}^d)}^2 &\le \frac{1}{|\delta_1|}\int_{\mathbb{R}^{d-1}}\left(\int_{\mathbb{R}}|u(y,\theta)|\,\mathrm{d}\theta\right)^2\left(\int_{\mathbb{R}}|v(y,\theta)|^2\,\mathrm{d}\theta\right)\mathrm{d}y\\ &\le \frac{\pi}{|\delta_1|}\int_{\mathbb{R}^{d-1}}\left(\int_{\mathbb{R}}(1+\theta^2)\,|u(y,\theta)|^2\,\mathrm{d}\theta\right)\left(\int_{\mathbb{R}}|v(y,\theta)|^2\,\mathrm{d}\theta\right)\mathrm{d}y\\ &\le \frac{\pi}{|\delta_1|}\left(\sup_{y\in\mathbb{R}^{d-1}}\int_{\mathbb{R}}(1+\theta^2)\,|u(y,\theta)|^2\,\mathrm{d}\theta\right)\|v\|_{L^2(\mathbb{R}^d)}^2\,.\end{aligned}$$

Since $k_0-1>(d-1)/2$, we have

$$|u(y,\theta)|^2\le C\sum_{|\alpha|\le k_0-1}\int_{\mathbb{R}^{d-1}}|\partial_y^\alpha u(y,\theta)|^2\,\mathrm{d}y\,,$$

by Sobolev's inequality, and we thus get (with a possibly larger constant C)

$$\|\mathbb{F}_{\rm pul}(u,v)\|_{L^2(\mathbb{R}^d)}\le C\,\|u\|_{\Gamma^{k_0}(\mathbb{R}^d)}\,\|v\|_{L^2(\mathbb{R}^d)}\,. \tag{6.7}$$

The "symmetric" inequality

$$\|\mathbb{F}_{\rm pul}(u,v)\|_{L^2(\mathbb{R}^d)}\le C\,\|u\|_{L^2(\mathbb{R}^d)}\,\|v\|_{\Gamma^{k_0}(\mathbb{R}^d)}\,, \tag{6.8}$$

is obtained by exchanging the roles of δ_1 and δ_3 as explained earlier.

2) Let us now estimate the H^k-norm of $\mathbb{F}_{\rm pul}(\partial_\theta u,\partial_\theta v)$ with $k\ge 2\,k_0+1$. We first apply the estimate (6.7) for the L^2-norm:

$$\|\mathbb{F}_{\rm pul}(\partial_\theta u,\partial_\theta v)\|_{L^2(\mathbb{R}^d)}\le C\,\|u\|_{\Gamma^{k_0+1}(\mathbb{R}^d)}\,\|v\|_{H^1(\mathbb{R}^d)}\le C\,\|u\|_{\Gamma^k(\mathbb{R}^d)}\,\|v\|_{\Gamma^k(\mathbb{R}^d)}\,.$$

Using Plancherel's Theorem, it is sufficient to estimate the k-th derivatives of $\mathbb{F}_{\rm pul}(\partial_\theta u,\partial_\theta v)$ in order to estimate all derivatives of order less than k. Let us therefore consider a multiinteger α of length k, and apply the Leibniz formula (this is justified because the differentiation formula (6.3) holds not only for the θ-derivative but also for the y-derivatives):

$$\partial^\alpha\,\mathbb{F}_{\rm pul}(\partial_\theta u,\partial_\theta v)=\sum_{\beta\le\alpha}\star\,\mathbb{F}_{\rm pul}(\partial^\beta\,\partial_\theta u,\partial^{\alpha-\beta}\,\partial_\theta v)\,, \tag{6.9}$$

where $\star$ denotes harmless numerical coefficients, and $\partial^\beta,\partial^{\alpha-\beta}$ stand for possibly mixed y,θ derivatives.

We begin with the extreme terms in (6.9). If $|\beta|=0$, we need to estimate the term $\mathbb{F}_{\rm pul}(\partial_\theta u,\partial^\alpha\,\partial_\theta v)$ which, using (6.4), we write as

$$-\frac{1}{\delta_3}\,\partial_\theta u\,\partial^\alpha v-\frac{\delta_1}{\delta_3}\,\mathbb{F}_{\rm pul}(\partial^2_{\theta\theta}u,\partial^\alpha v)\,.$$

We get the estimate

$$\|\mathbb{F}_{\rm pul}(\partial_\theta u,\partial^\alpha\,\partial_\theta v)\|_{L^2(\mathbb{R}^d)}\le C\,\|\partial_\theta u\|_{L^\infty(\mathbb{R}^d)}\,\|v\|_{H^k(\mathbb{R}^d)}+C\,\|u\|_{\Gamma^{k_0+2}(\mathbb{R}^d)}\,\|v\|_{H^k(\mathbb{R}^d)}\,,$$

where we have used (6.7). Since $k>d/2+1$ (this follows from the assumption $k\ge 2\,k_0+1$), we can apply Sobolev's inequality and get

$$\|\mathbb{F}_{\rm pul}(\partial_\theta u,\partial^\alpha\,\partial_\theta v)\|_{L^2(\mathbb{R}^d)}\le C\,\|u\|_{\Gamma^k(\mathbb{R}^d)}\,\|v\|_{\Gamma^k(\mathbb{R}^d)}\,.$$

The second extreme term $\mathbb{F}_{\text{pul}}(\partial^\alpha \partial_\theta u, \partial_\theta v)$ is dealt with in the same way.

Using the assumption $k \geq 2\,k_0 + 1$, we verify that for $\beta \leq \alpha$, and $\beta \neq 0$, $\beta \neq \alpha$, one of the following two properties is satisfied

$$(|\beta| \geq 1 \quad \text{and} \quad 1 + |\beta| \leq k - k_0) \quad \text{or} \quad (|\alpha| - |\beta| \geq 1 \quad \text{and} \quad |\beta| \geq k_0 + 1)\,.$$

In the first case, we use (6.7) and get

$$\|\mathbb{F}_{\text{pul}}(\partial^\beta \partial_\theta u, \partial^{\alpha-\beta} \partial_\theta v)\|_{L^2(\mathbb{R}^d)} \leq C\,\|\partial^\beta \partial_\theta u\|_{\Gamma^{k_0}(\mathbb{R}^d)}\,\|\partial^{\alpha-\beta} \partial_\theta v\|_{L^2(\mathbb{R}^d)} \leq C\,\|u\|_{\Gamma^k(\mathbb{R}^d)}\,\|v\|_{\Gamma^k(\mathbb{R}^d)}\,,$$

and the second case is dealt with in a symmetric way (using (6.8) rather than (6.7)).

3) It remains to estimate the L^2-norm of $\theta^k\,\mathbb{F}_{\text{pul}}(\partial_\theta u, \partial_\theta v)$. We write (use $\delta_3 - \delta_1 = 1$ again)

$$\theta = \delta_3\,(\theta + \delta_1\,s) - \delta_1\,(\theta + \delta_3\,s)\,,$$

which gives

$$\theta^k\,\mathbb{F}_{\text{pul}}(\partial_\theta u, \partial_\theta v) = \sum_{j=0}^{k} \star\,\mathbb{F}_{\text{pul}}(\theta^j\,\partial_\theta u, \theta^{k-j}\,\partial_\theta v)\,, \tag{6.10}$$

with, again, harmless binomial coefficients that are denoted by $\star$. Let us first consider the extreme terms in the latter sum and focus on $\mathbb{F}_{\text{pul}}(\partial_\theta u, \theta^k\,\partial_\theta v)$. We write

$$\theta^k\,\partial_\theta v = \partial_\theta(\theta^k\,v) - k\,\theta^{k-1}\,v\,,$$

and use the property (6.4) to get

$$\mathbb{F}_{\text{pul}}(\partial_\theta u, \theta^k\,\partial_\theta v) = -\frac{1}{\delta_3}\,\partial_\theta u\,(\theta^k\,v) - \frac{\delta_1}{\delta_3}\,\mathbb{F}_{\text{pul}}(\partial^2_{\theta\theta} u, \theta^k\,v) - k\,\mathbb{F}_{\text{pul}}(\partial_\theta u, \theta^{k-1}\,v)\,.$$

The L^2-estimate follows from (6.7) and from the Sobolev imbedding Theorem:

$$\|\mathbb{F}_{\text{pul}}(\partial_\theta u, \theta^k\,\partial_\theta v)\|_{L^2(\mathbb{R}^d)} \leq C\,\|u\|_{\Gamma^k(\mathbb{R}^d)}\,\|v\|_{\Gamma^k(\mathbb{R}^d)}\,.$$

The estimate for $\mathbb{F}_{\text{pul}}(\theta^k\,\partial_\theta u, \partial_\theta v)$ is similar.

For $1 \leq j \leq k-1$, we observe again that there holds either $j \leq k - k_0 - 1$ or $j \geq k_0 + 1$, so that we can directly estimate all intermediate terms in the sum (6.10) by applying either (6.7) or (6.8). The proof of Proposition 6.2 is now complete. □

As an immediate corollary of the proof we have

Corollary 6.3. *Let k_0 denote the smallest integer satisfying $k_0 > (d+1)/2$. Then for all $k \geq 2\,k_0$, there exists a constant C_k satisfying*

$$\forall\,u, v \in \Gamma^k(\mathbb{R}^d)\,, \quad \|\mathbb{F}_{\text{pul}}(u, v)\|_{\Gamma^k(\mathbb{R}^d)} \leq C_k\,\|u\|_{\Gamma^k(\mathbb{R}^d)}\,\|v\|_{\Gamma^k(\mathbb{R}^d)}\,. \tag{6.11}$$

6.3 The iteration scheme

In view of the boundedness property proved in Proposition 6.2, Equation (6.5) is a semilinear perturbation of the Burgers equation (the transport term $\mathbf{w} \cdot \nabla_y$ is harmless), and it is absolutely not surprising that we can solve (6.5) by using the standard energy method on a fixed point iteration. This well-posedness result can be summarized in the following Theorem.

Theorem 6.4. *Let k_0 be defined as in Proposition 6.2, and let $k \geq 2\,k_0 + 1$. Then for all $R > 0$, there exists a time $T(k, R) > 0$ such that for all data $a_0 \in \Gamma^k(\mathbb{R}^d)$ satisfying $\|a_0\|_{\Gamma^k(\mathbb{R}^d)} \leq R$, there exists a unique solution $a \in \mathcal{C}([0, T]; \Gamma^k(\mathbb{R}^d))$ to the Cauchy problem:*

$$\begin{cases} \partial_t a + \mathbf{w} \cdot \nabla_y a + c\, a\, \partial_\theta a + \mu\, \mathbb{F}_{\text{pul}}(\partial_\theta a, \partial_\theta a) = 0\,, \\ a|_{t=0} = a_0\,. \end{cases}$$

Of course, one can also incorporate a nonzero source term $g \in L^2([0, T_0]; \Gamma^k(\mathbb{R}^d))$ in the equation, and obtain a unique solution $a \in \mathcal{C}([0, T]; \Gamma^k(\mathbb{R}^d))$ on a time interval $[0, T]$ with $T \leq T_0$. We omit the details that are completely standard.

Proof. We follow the standard energy method for quasilinear symmetric systems, see for instance [BGS07, chapter 10], and solve the Cauchy problem by the iteration scheme

$$\begin{cases} \partial_t a^{n+1} + \mathbf{w} \cdot \nabla_y a^{n+1} + c\, a^n\, \partial_\theta a^{n+1} + \mu\, \mathbb{F}_{\text{pul}}(\partial_\theta a^n, \partial_\theta a^n) = 0\,, \\ a^{n+1}|_{t=0} = a_{0,n+1}\,, \end{cases}$$

where $(a_{0,n})$ is a sequence of, say, Schwartz functions that converges towards a_0 in $\Gamma^k(\mathbb{R}^d)$, and the scheme is initialized with the choice $a^0 \equiv a_{0,0}$. Given the radius R for the ball in $\Gamma^k(\mathbb{R}^d)$, we can choose some time $T > 0$, that only depends on R and k, such that the sequence (a^n) is bounded in $\mathcal{C}([0, T]; \Gamma^k(\mathbb{R}^d))$. The uniform bound in $\mathcal{C}([0, T]; H^k(\mathbb{R}^d))$ is proved by following the exact same ingredients as in the case of the Burgers equation, and the weighted L^2 bound is proved by computing

$$\partial_t(\theta^k\, a^{n+1}) + \mathbf{w} \cdot \nabla_y(\theta^k\, a^{n+1}) + c\, a^n\, \partial_\theta(\theta^k\, a^{n+1}) = -\mu\, \theta^k\, \mathbb{F}_{\text{pul}}(\partial_\theta a^n, \partial_\theta a^n) + k\, c\, a^n\, (\theta^{k-1}\, \partial_\theta a^{n+1})\,,$$

and performing the usual L^2-estimate for the transport equation.

It remains to show that the iteration contracts in the $\mathcal{C}([0, T]; L^2(\mathbb{R}^d))$-topology for T small enough. This is proved by defining $r^{n+1} := a^{n+1} - a^n$ and computing

$$\partial_t r^{n+1} + \mathbf{w} \cdot \nabla_y r^{n+1} + c\, a^n\, \partial_\theta r^{n+1} = -c\, r^n\, \partial_\theta a^n - \mu\, \mathbb{F}_{\text{pul}}(\partial_\theta r^n, \partial_\theta a^n) - \mu\, \mathbb{F}_{\text{pul}}(\partial_\theta a^{n-1}, \partial_\theta r^n)\,.$$

The error terms on the right hand-side are written as

$$\begin{aligned} \mathbb{F}_{\text{pul}}(\partial_\theta r^n, \partial_\theta a^n) &= -\frac{1}{\delta_1}\, r^n\, \partial_\theta a^n - \frac{\delta_3}{\delta_1}\, \mathbb{F}_{\text{pul}}(r^n, \partial^2_{\theta\theta} a^n)\,, \\ \mathbb{F}_{\text{pul}}(\partial_\theta a^{n-1}, \partial_\theta r^n) &= -\frac{1}{\delta_3}\, r^n\, \partial_\theta a^{n-1} - \frac{\delta_1}{\delta_3}\, \mathbb{F}_{\text{pul}}(\partial^2_{\theta\theta} a^{n-1}, r^n)\,, \end{aligned}$$

and we then apply the L^2-estimates (6.7) and (6.8). This gives contraction of the iteration scheme in $\mathcal{C}([0, T]; L^2(\mathbb{R}^d))$ and, passing to the limit, we obtain a solution $a \in L^\infty([0, T]; \Gamma^k(\mathbb{R}^d))$ to the Cauchy problem. Continuity in $\Gamma^k(\mathbb{R}^d)$ is recovered by the same arguments as in [BGS07, chapter 10], using the norm in $\Gamma^k(\mathbb{R}^d)$ rather than the H^k-norm. We feel free to skip the details. ☐

We do not claim that the regularity assumption in Theorem 6.4 is optimal as far as local existence of smooth solutions is concerned, but it is sufficient for our purpose. The global existence of weak and/or smooth solutions is, of course, a wide open problem. Numerical simulations reported in [MR84] tend to indicate that finite time breakdown of smooth solutions should be expected, and Proposition 6.2 is clearly a first step towards a rigorous justification of this fact. We postpone the study of such finite time breakdown to a future work.

6.4 Construction of the leading profile

Corollary 6.5 below is based on Theorem 6.4 and is entirely similar to Corollary 3.5 for the wavetrain problem. The only difference is that we restrict here to some finite regularity since the estimate provided by Theorem 6.4 is not tame. Let us recall that the smoothness assumption for the source term G in (1.1) was made in Theorem 1.11.

Corollary 6.5. *With $K_0, K_1 \in \mathbb{N}$ and G as in Theorem 1.11, there exists $0 < T \leq T_0$, and there exists a unique*

$$a \in \cap_{\ell=0}^{K_0} \mathcal{C}^\ell((-\infty, T]; \Gamma^{K_1-1-\ell}(\mathbb{R}^d_{y,\theta}))\,,$$

solution to (6.13), (6.14), with $a|_{t<0} = 0$. Furthermore, the integral of a with respect to the variable $\theta \in \mathbb{R}$ vanishes.

Proof. The proof is rather straightforward. Since $G \in \mathcal{C}^0((-\infty, T]; \Gamma^{K_1}(\mathbb{R}^d))$, Theorem 6.4 yields a unique solution $a \in \mathcal{C}^0((-\infty, T]; \Gamma^{K_1-1}(\mathbb{R}^d))$ to (6.13), (6.14), with $a|_{t<0} = 0$. Furthermore, (6.13) automatically yields $a \in \mathcal{C}^1((-\infty, T]; \Gamma^{K_1-2}(\mathbb{R}^d))$ thanks to Proposition 6.2 and the fact that $\Gamma^{K_1-2}(\mathbb{R}^d)$ is an algebra. Then the standard bootstrap argument yields

$$a \in \cap_{\ell=0}^{K_0} \mathcal{C}^\ell((-\infty, T]; \Gamma^{K_1-1-\ell}(\mathbb{R}^d_{y,\theta}))\,,$$

by differentiating (6.13) sufficiently many times with respect to time.

Let us observe that the requirement $K_1 - K_0 - 1 \geq 2\,k_0 + 1$ in Theorem 1.11 is used here to apply Proposition 6.2 for the term $\partial_{\theta_0} Q_{\text{pul}}[a, a]$ and its successive time derivatives.

Thanks to the property $a \in \mathcal{C}^1(\Gamma^{K_1-2})$, a is integrable with respect to $\theta \in \mathbb{R}$, and integration of (6.13) yields

$$X_{\text{Lop}}\, \underline{a} = 0\,, \quad \underline{a}|_{t<0} = 0\,,$$

where $\underline{a}$ denotes the integral of a with respect to θ. Hence $\underline{a}$ is identically zero. $\square$

Corollary 6.6. *Up to restricting $T > 0$ in Corollary 6.5, for all $m \in \mathcal{I}$, there exists a unique solution*

$$\sigma_m \in \cap_{\ell=0}^{K_0} \mathcal{C}^\ell((-\infty, T]; \Gamma^{K_0-\ell}(\mathbb{R}^d_+ \times \mathbb{R}_\theta))$$

to (5.19) with $\sigma_m|_{t<0} = 0$ and $\sigma_m|_{x_d=0} = \mathfrak{e}_m\, a$, where the real number $\mathfrak{e}_m$ is defined by $e_m = \mathfrak{e}_m\, r_m$. Furthermore, each σ_m has zero integral with respect to the variable $\theta_m \in \mathbb{R}$.

Proof. From Corollary 6.5, we get $a \in \Gamma^{K_0}((-\infty, T]_t \times \mathbb{R}^d_{y,\theta})$, with $K_0 > 1+(d+1)/2$ (here we use $K_1 - 1 \geq K_0$). Then we solve the Burgers equation (5.19) with prescribed boundary condition $\sigma_m|_{x_d=0} = \mathfrak{e}_m\, a$. The theory is similar to that in the standard Sobolev space H^{K_0}, and we feel free to use well-posedness in this weighted Sobolev space framework. This yields a solution

$$\sigma_m \in \cap_{\ell=0}^{K_0} \mathcal{C}^\ell((-\infty, T]; \Gamma^{K_0-\ell}(\mathbb{R}^d_+ \times \mathbb{R}_\theta))$$

to (5.19) that vanishes in the past. Integration of (5.19) with respect to θ_m shows that the integral of σ_m with respect to the variable $\theta_m \in \mathbb{R}$ satisfies

$$\partial_t \underline{\sigma_m} + \mathbf{v}_m \cdot \nabla_x \underline{\sigma_m} = 0\,, \quad \underline{\sigma_m}|_{x_d=0} = 0\,,$$

and therefore vanishes. □

Remark 6.7. *We observe that there has been a rather big loss of regularity in passing from the trace a to the interior functions σ_m. This is due to the following fact: the trace a is obtained by solving a Cauchy problem, where tangential regularity with respect to the time slices $\{t = \text{constant}\}$ is propagated. However, constructing a local in time smooth solution to (5.19) requires a full regularity for the trace a, that is, regularity of both (y, θ) and time partial derivatives. This is the reason why we also need to control time derivatives of a, which means controlling time derivatives of $\partial_{\theta_0} Q_{\text{pul}}[a, a]$. In Corollary 6.5, we have considered sufficiently smooth initial data so that such time derivatives are controlled by an easy application of Proposition 6.2. Again, we do not aim at an optimal regularity result.*

6.5 Extension to more general $N \times N$ systems.

The extension of the definitions ($\mathcal{V}_F, \mathcal{V}_H, E_P, E_Q, R_\infty$, etc.) and results for pulses in the strictly hyperbolic 3×3 case to the strictly hyperbolic $N \times N$ case is straightforward. We first show that the leading profile $\mathcal{U}_0$ reads

$$\mathcal{U}_0(t, x, \theta_0, \xi_d) = \sum_{m \in \mathcal{I}} \sigma_m(t, x, \theta_0 + \underline{\omega}_m\, \xi_d)\, r_m\,, \tag{6.12}$$

with functions σ_m that are expected to decay at infinity with respect to θ_m. Moreover, the σ_m's satisfy (5.19) in the domain $\{x_d > 0\}$, and (5.4)(a) yields

$$\forall\, m \in \mathcal{I}\,, \quad \sigma_m(t, y, 0, \theta_0)\, r_m = a(t, y, \theta_0)\, e_m\,,$$

for a suitable function a.

The existence of a bounded corrector $\mathcal{U}_2$ solution to (5.2)(c) implies that the first corrector reads

$$\begin{aligned} \mathcal{U}_1(t, x, \theta_0, \xi_d) = &\sum_{m \in \mathcal{O}} -\int_{\xi_d}^{+\infty} \ell_m\, \mathcal{F}_0(t, x, \theta_0 + \underline{\omega}_m\, (\xi_d - X), X)\, \mathrm{d}X\, r_m \\ &+ \sum_{m \in \mathcal{I}} \left(\tau_m(t, x, \theta_0 + \underline{\omega}_m\, \xi_d) - \int_{\xi_d}^{+\infty} \ell_m\, \mathcal{F}_0(t, y, 0, \theta_0 + \underline{\omega}_m\, (\xi_d - X), X)\, \mathrm{d}X \right) r_m\,. \end{aligned}$$

Plugging these expressions in (5.4)(b), we derive the following amplitude equation that governs the evolution of a on the boundary:

$$\upsilon\, \partial_{\theta_0}(a^2) - X_{\text{Lop}} a + \partial_{\theta_0} Q_{\text{pul}}[a, a] = \underline{b}\, \partial_{\theta_0} G\,, \tag{6.13}$$

with υ and X_{Lop} defined in (2.23), and Q_{pul} defined by:

(6.14)

$$\begin{aligned} Q_{\text{pul}}[a, \widetilde{a}](\theta_0) := &\sum_{m \in \mathcal{O}} \underline{b}\, B\, r_m \sum_{\substack{m_1 < m_2 \\ m_1, m_2 \in \mathcal{I}}} \ell_m\, E_{m_1, m_2} \int_0^{+\infty} \partial_{\theta_0} a(\theta_0 + (\underline{\omega}_{m_1} - \underline{\omega}_m)\, X)\, \widetilde{a}(\theta_0 + (\underline{\omega}_{m_2} - \underline{\omega}_m)\, X)\, \mathrm{d}X \\ &+ \sum_{m \in \mathcal{O}} \underline{b}\, B\, r_m \sum_{\substack{m_1 < m_2 \\ m_1, m_2 \in \mathcal{I}}} \ell_m\, E_{m_2, m_1} \int_0^{+\infty} a(\theta_0 + (\underline{\omega}_{m_1} - \underline{\omega}_m)\, X)\, \partial_{\theta_0} \widetilde{a}(\theta_0 + (\underline{\omega}_{m_2} - \underline{\omega}_m)\, X)\, \mathrm{d}X \end{aligned}$$

with E_{m_1,m_2} as in (2.25).

As in the wavetrain case some care is needed in the extension to hyperbolic systems with constant multiplicities. Instead of introducing bases $\{\ell_{m,k}\}$, $\{r_{m,k}\}$, $m=1,\dots,M$, $k=1,\dots,\nu_{k_m}$ of left and right eigenvectors, we now define the spaces $\mathcal{V}_F$ and $\mathcal{V}_H$ and operators E_P, E_Q, R_∞ as follows.

Definition 6.8. *a) Define the set of "constituent functions" $\mathcal{C}=\cup_{m=1}^M\mathcal{C}^m$, where $\mathcal{C}^m$ is the set of R^N- valued functions of (t,x,θ_0,ξ_d) of the form $F(t,x,\theta_0+\underline{\omega}_m\xi_d)$, where $F(t,x,\theta_m)$ is C^1 and decays with its first-order partials at the rate $O(\langle\theta\rangle^{-2})$ uniformly with respect to (t,x). Setting $\mathcal{V}_{\mathcal{C}}=\oplus_{m=1}^M\mathcal{C}^m$, we can write any $F\in\mathcal{V}_{\mathcal{C}}$ as $F=\sum_{m=1}^M F^m$ with $F^m\in\mathcal{C}^m$.*

b) Define $\mathcal{V}_F$ to be the space of $\mathbb{R}^N$ valued functions of (t,x,θ_0,ξ_d) that may be written as a finite sum of terms of the form

$$F,\quad \mathcal{B}_\alpha(G,H),\quad \mathcal{T}_\beta(K,L,M), \tag{6.15}$$

where F, $G,\dots,M$ lie in $\mathcal{V}_{\mathcal{C}}$, $\mathcal{B}_\alpha:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is any bilinear map, and $\mathcal{T}_\beta:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is any trilinear map.

c) Define a transversal interaction integral *to be a function $I^i_{m,l}$ of the form*

$$I^i_{l,m}(t,x,\theta_0,\xi_d)=\int_{+\infty}^{\xi_d}A_d(0)^{-1}Q_i\mathcal{B}_\gamma(F^l,G^m)(t,x,\theta_0+\underline{\omega}_i(\xi_d-s),s)ds, \tag{6.16}$$

where i,l,m lie in $\{1,\dots,M\}$ and are mutually distinct, $\mathcal{B}_\gamma:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is any bilinear map, and $F^l\in\mathcal{C}^l$, $G^m\in\mathcal{C}^m$ are required to decay with their first-order partials at the rate $O(\langle\theta\rangle^{-3})$ uniformly with respect to (t,x).

d) Define $\mathcal{V}_H$ to be the space of $\mathbb{R}^N$-valued functions of (t,x,θ_0,ξ_d) that may be written as the sum of an element of $\mathcal{V}_F$ plus a finite sum of terms of the form

$$I^i_{l,m}(t,x,\theta_0,\xi_d)\ \text{or}\ \mathcal{B}_\delta(A^j,J^n_{p,q}(t,x,\theta_0,\xi_d)), \tag{6.17}$$

where $I^i_{l,m}$ and $J^n_{p,q}$ are transversal interaction integrals, $\mathcal{B}_\delta:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is any bilinear map, and $A^j\in\mathcal{C}^j$ is required to decay with its first-order partials at the rate $O(\langle\theta\rangle^{-3})$ uniformly with respect to (t,x).

e) For $H\in\mathcal{V}_H$ we can write $H=H_F+H_I$, where $H_F\in\mathcal{V}_F$ and $H_I\notin\mathcal{V}_F$ is a finite sum of terms of the form (6.17). The "constituent functions" of H include those of H_F as well as the functions like A^j, F^l, G^m as in (6.16), (6.17) which constitute H_I.

f) With these definitions of $\mathcal{V}_F$, $\mathcal{V}_H$, and "constituent functions", the subspaces $\mathcal{V}^k_F$ and $\mathcal{V}^k_H$ may be defined just as in Definition 5.8.

Definition 6.9 (E_P,E_Q,R_∞)**.** *For $H\in\mathcal{V}_H$ define averaging operators*

$$\begin{aligned}E_QH(t,x,\theta_0,\xi_d)&=\sum_{j=1}^M\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T Q_jH(x,\theta_0+\underline{\omega}_j(\xi_d-s),s)ds\right)\\ E_PH(t,x,\theta_0,\xi_d)&=\sum_{j=1}^M\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T P_jH(x,\theta_0+\underline{\omega}_j(\xi_d-s),s)ds\right).\end{aligned} \tag{6.18}$$

For $H\in\mathcal{V}_H$ such that $E_QH=0$ define the solution operator

$$R_\infty H(t,x,\theta_0,\xi_d)=\sum_{j=1}^M\left(\int_{+\infty}^{\xi_d}A_d(0)^{-1}Q_jH(t,x,\theta_0+\underline{\omega}_j(\xi_d-s),s)ds\right). \tag{6.19}$$

With these definitions Propositions 5.6 and 5.7 hold with the obvious minor changes. For example, in Proposition 5.7(c) we now have

$$E_P\mathcal{U} = \sum_{m=1}^{M} \mathcal{U}^m, \text{ where } \mathcal{U}^m \in \mathcal{C}^m \text{ and } P_m\mathcal{U}^m = \mathcal{U}^m. \tag{6.20}$$

The construction of profiles is carried out assuming $\mathcal{U}_2 \in C^1_b$ and that $\mathcal{U}_0, \mathcal{U}_1$ satisfy (5.17), where (5.17)(a) is modified to

$$\mathcal{U}_0 = \sum_{m=1}^{M} \mathcal{U}_0^m, \;\; \mathcal{U}_0^m = P_m\mathcal{U}_0^m \in \mathcal{C}^m \cap \mathcal{V}_F^{K_0}, \;\; \int_{-\infty}^{+\infty} \mathcal{U}_0^m(t,x,\theta_m)d\theta_m = 0. \tag{6.21}$$

Again the interior leading profile equations for the $Q_m\mathcal{U}_0^m$ take the form (4.17), which allows the moment zero property of the $\mathcal{U}_0^m$ to be deduced from that of $a(t,y,\theta_0)$ as before. The solvability in Γ^k spaces of (4.17) with boundary conditions (4.21) follows from Lemma 4.2 via L^2 estimates proved in the standard way.[13] One finds as before that $\mathcal{U}_0$ is purely incoming (4.20).

The consequence (4.16) of the conservative structure assumption is used again in step 2 of the profile construction when performing the integral that now replaces the integral $\int_{+\infty}^{\xi_d}(\cdot)ds$ of the second line of (5.24). This integral now reads

$$\sum_{k\neq i} \int_{+\infty}^{\xi_d} A_d(0)^{-1}Q_i\mathcal{M}(\mathcal{U}_0^k,\mathcal{U}_0^k)(t,x,\theta_0 + \underline{\omega}_i(\xi_d - s), s)ds. \tag{6.22}$$

The replacements for the other lines of (5.24) are treated essentially as before; the last line now yields transversal interaction integrals of the form (6.16).

Parallel to (4.22) the nonlocal operator in the equation for a now has the form

$$\begin{aligned} Q_{\rm pul}[a,\widetilde{a}](\theta_0) := &\sum_{m\in\mathcal{O}} \sum_{\substack{m_1<m_2\\ m_1,m_2\in\mathcal{I}}} \underline{b}\, BA_d(0)^{-1}Q_m E_{m_1,m_2} \int_0^{+\infty} \partial_{\theta_0} a(\theta_0 + (\underline{\omega}_{m_1} - \underline{\omega}_m)\,X)\,\widetilde{a}(\theta_0 + (\underline{\omega}_{m_2} - \underline{\omega}_m)\,X)\,{\rm d}X \\ &+ \sum_{m\in\mathcal{O}} \sum_{\substack{m_1<m_2\\ m_1,m_2\in\mathcal{I}}} \underline{b}\, BA_d(0)^{-1}Q_m E_{m_2,m_1} \int_0^{+\infty} a(\theta_0 + (\underline{\omega}_{m_1} - \underline{\omega}_m)\,X)\,\partial_{\theta_0}\widetilde{a}(\theta_0 + (\underline{\omega}_{m_2} - \underline{\omega}_m)\,X)\,{\rm d}X \end{aligned} \tag{6.23}$$

with E_{m_1,m_2} as in (2.25).

Repetition of step 6 of the profile construction yields $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{U}_2$ with the same regularity and decay properties as before.

[13] Such an argument is given in detail in Propositions 3.5 and 3.6 of [CW13].

A Example: the two-dimensional isentropic Euler equations

In this first appendix, we discuss how our main results apply to the two-dimensional Euler equations in a fixed half-plane. Once again, we refer to [MR83, AM87, WY14] for the derivation of such amplified high frequency expansions in various free boundary problems. Our discussion here will mainly deal with the verification of the small divisors condition, that is, Assumption 1.9. In quasilinear form, the isentropic Euler equations read

$$
\begin{cases}
\partial_t v + u_1\,\partial_1 v + u_2\,\partial_2 v - v\,(\partial_1 u_1 + \partial_2 u_2) = 0\,, \\
\partial_t u_1 + u_1\,\partial_1 u_1 + u_2\,\partial_2 u_1 - \dfrac{c(v)^2}{v}\,\partial_1 v = 0\,, \\
\partial_t u_2 + u_1\,\partial_1 u_2 + u_2\,\partial_2 u_2 - \dfrac{c(v)^2}{v}\,\partial_2 v = 0\,,
\end{cases}
\tag{A.1}
$$

where $v > 0$ denotes the specific volume of the fluid, $c(v) > 0$ denotes the sound speed and (u_1, u_2) the velocity field. We consider a fixed reference volume $\underline{v}$, and a fixed velocity field $(0, \underline{u})$ with

$$
\underline{v} > 0\,, \quad 0 < \underline{u} < \underline{c} := c(\underline{v})\,,
$$

which corresponds to an *incoming subsonic* fluid. The above theory applies when looking for WKB solutions to (A.1) of the form

$$
\begin{pmatrix} v \\ u_1 \\ u_2 \end{pmatrix}_\varepsilon \sim \begin{pmatrix} \underline{v} \\ 0 \\ \underline{u} \end{pmatrix} + \varepsilon \sum_{n \geq 0} \varepsilon^n\, \mathcal{U}_n \left(t, x, \frac{\Phi(t,x)}{\varepsilon} \right)\,,
$$

provided that the linearization of (A.1) (with appropriate boundary conditions) at the reference state $(\underline{v}, 0, \underline{u})$ satisfies all assumptions of Section 1. Let us therefore consider the linearization of (A.1) at $(\underline{v}, 0, \underline{u})$, which corresponds, in the notation of Section 1, to

$$
A_1(0) := \begin{pmatrix} 0 & -\underline{v} & 0 \\ -\underline{c}^2/\underline{v} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad A_2(0) := \begin{pmatrix} \underline{u} & 0 & -\underline{v} \\ 0 & \underline{u} & 0 \\ -\underline{c}^2/\underline{v} & 0 & \underline{u} \end{pmatrix}\,,
$$

$$
\mathrm{d}A_1(0) \cdot \begin{pmatrix} v \\ u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} u_1 & -v & 0 \\ (-2\,\underline{c}\,\underline{c}'/\underline{v} + \underline{c}^2/\underline{v}^2)\,v & u_1 & 0 \\ 0 & 0 & u_1 \end{pmatrix}\,,
$$

$$
\mathrm{d}A_2(0) \cdot \begin{pmatrix} v \\ u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} u_2 & 0 & -v \\ 0 & u_2 & 0 \\ (-2\,\underline{c}\,\underline{c}'/\underline{v} + \underline{c}^2/\underline{v}^2)\,v & 0 & u_2 \end{pmatrix}\,,
$$

where $\underline{c}'$ stands for $c'(\underline{v})$. The operator $\partial_t + A_1(0)\,\partial_1 + A_2(0)\,\partial_2$ is strictly hyperbolic with eigenvalues

$$
\lambda_1(\xi_1, \xi_2) := \underline{u}\,\xi_2 - \underline{c}\,\sqrt{\xi_1^2 + \xi_2^2}\,, \quad \lambda_2(\xi_1, \xi_2) := \underline{u}\,\xi_2\,, \quad \lambda_3(\xi_1, \xi_2) := \underline{u}\,\xi_2 + \underline{c}\,\sqrt{\xi_1^2 + \xi_2^2}\,,
$$

so Assumptions 1.1 and 1.2 are satisfied. There are two incoming characteristics and one outgoing characteristic, so the matrix B encoding the boundary conditions for the linearized equations should be a 2×3 matrix of maximal rank. One possible choice for B is made precise below. The hyperbolic region $\mathcal{H}$ is given by

$$
\mathcal{H} = \left\{ (\tau, \eta) \in \mathbb{R} \times \mathbb{R} \,/\, |\tau| > \sqrt{\underline{c}^2 - \underline{u}^2}\,|\eta| \right\}\,.
$$

We focus on the verification of Assumption 1.9. For concreteness, let $(\tau, \eta) \in \mathcal{H}$ with $\tau > 0$ and $\eta > 0$, and consider the boundary phase

$$\varphi_0(t, x_1) := \tau\, t + \eta\, x_1 \,.$$

The three (distinct) eigenvalues of $\mathcal{A}(\tau, \eta)$ are

$$\omega_1 := \frac{\underline{u}\, \tau - \underline{c}\, \sqrt{\tau^2 - (\underline{c}^2 - \underline{u}^2)\, \eta^2}}{\underline{c}^2 - \underline{u}^2} \,, \quad \omega_2 := \frac{\underline{u}\, \tau + \underline{c}\, \sqrt{\tau^2 - (\underline{c}^2 - \underline{u}^2)\, \eta^2}}{\underline{c}^2 - \underline{u}^2} \,, \quad \omega_3 := -\frac{\tau}{\underline{u}} \,,$$

and they satisfy

$$\tau + \lambda_1(\eta, \omega_1) = \tau + \lambda_1(\eta, \omega_2) = \tau + \lambda_2(\eta, \omega_3) = 0 \,.$$

The associated group velocities are

$$\mathbf{v}_1 := \frac{1}{\tau + \underline{u}\, \omega_1} \begin{pmatrix} -\underline{c}^2\, \eta \\ \underline{c}\, \sqrt{\tau^2 - (\underline{c}^2 - \underline{u}^2)\, \eta^2} \end{pmatrix} \,, \quad \mathbf{v}_2 := \frac{1}{\tau + \underline{u}\, \omega_2} \begin{pmatrix} -\underline{c}^2\, \eta \\ -\underline{c}\, \sqrt{\tau^2 - (\underline{c}^2 - \underline{u}^2)\, \eta^2} \end{pmatrix} \,, \quad \mathbf{v}_3 := \begin{pmatrix} 0 \\ \underline{u} \end{pmatrix} \,,$$

hence the phase φ_2 is outgoing while φ_1, φ_3 are incoming (as in the framework of Paragraph 2.2). The nonresonance condition in Assumption 1.9, that is,

$$\forall\, \alpha \in \mathbb{Z}^3 \setminus \mathbb{Z}^{3;1} \,, \quad \det L(\mathrm{d}(\alpha \cdot \Phi)) \neq 0 \,,$$

holds if and only if the (dimensionless) quantity

$$\frac{\underline{u}}{\underline{c}} \sqrt{1 - (\underline{c}^2 - \underline{u}^2)\, \frac{\eta^2}{\tau^2}} \,,$$

is not a rational number, as follows from a straightforward calculation. We thus introduce a positive irrational parameter ζ defined by

$$\frac{\underline{u}}{\underline{c}} \sqrt{\tau^2 - (\underline{c}^2 - \underline{u}^2)\, \eta^2} = \zeta\, \tau \,. \tag{A.2}$$

Our choice of parameters gives $\zeta \in (0, 1)$. For all $\alpha_1, \alpha_3 \in \mathbb{Z} \setminus \{0\}$, we compute

$$\begin{aligned} \det L(\mathrm{d}(\alpha_1\, \varphi_1 + \alpha_3\, \varphi_3)) &= -\alpha_1\, \alpha_3\, \underline{c}^2\, (\tau + \underline{u}\, \omega_1) \left[2\, \alpha_1\, (\eta^2 + \omega_1\, \omega_3) + \alpha_3\, (\eta^2 + \omega_3^2) \right] \\ &= \alpha_1\, \alpha_3\, (\tau + \underline{u}\, \omega_1)\, \frac{\underline{c}^4\, \tau^2}{\underline{u}^2\, (\underline{c}^2 - \underline{u}^2)}\, (\zeta - 1) \left(2\, \alpha_1\, \zeta + \alpha_3\, (\zeta + 1) \right) . \end{aligned}$$

It appears from the latter expression that the verification of Assumption 1.9 only depends on the arithmetic properties of the parameter ζ in (A.2). In particular, we have the following result.

Lemma A.1. *For all $\gamma > 1$, there exists a set of full measure $\mathcal{M}_\gamma$ in $(0, 1)$ such that for all $M \in \mathcal{M}_\gamma$ and $\underline{c} > 0$, if $\underline{u} = M\, \underline{c}$ and $\tau = \underline{c}\, \eta$, then Assumption 1.9 is satisfied for some constant $c > 0$.*

Of course, one could also fix the parameters $\underline{u}, \underline{c}$ and make τ vary in the hyperbolic region, and obtain a similar result (meaning that Assumption 1.9 would be satisfied except possibly for τ in a negligible set).

Proof. Choosing $\tau = \underline{c}\, \eta$ with $\eta > 0$, one has $(\tau, \eta) \in \mathcal{H}$, and the parameter ζ in (A.2) equals M^2 with $M := \underline{u}/\underline{c}$. Using the above expression for the determinant of $L(\mathrm{d}(\alpha_1\, \varphi_1 + \alpha_3\, \varphi_3))$, we see that the small divisors condition of Assumption 1.9 will be satisfied provided that there exists a constant $c > 0$ such that

$$\forall\,(p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}\,, \quad |p + q\,M^2| \ge c\,|(p,q)|^{-\gamma}\,. \tag{A.3}$$

For $c \in (0,1)$ and $\gamma > 1$, let us define

$$\mathcal{N}_c := \Big\{ m \in (0,1)\,/\,\exists\,(p,q) \in \mathbb{Z}^2\,, \quad q \ne 0 \quad \text{and} \quad |p + q\,m| < \frac{c}{|(p,q)|^\gamma} \Big\}\,.$$

The set $\mathcal{N}_c$ is the countable union, as p varies in $\mathbb{Z}$ and q in $\mathbb{Z}^*$, of intervals of width at most $2\,c/(|q|\,|(p,q)|^\gamma)$. Hence the Lebesgue measure of $\mathcal{N}_c$ is $O(c)$, which means that the intersection $\cap_{c>0}\mathcal{N}_c$ is negligible. Consequently, for every fixed $\gamma > 1$, the set of parameters M^2 such that (A.3) is satisfied for some constant $c > 0$ has full measure 1. The claim of Lemma A.1 follows. $\square$

We assume from now on that τ, η are fixed such that $\tau = \underline{c}\,\eta$, and the Mach number M is chosen in such a way that Assumption 1.9 is satisfied (which is some kind of a generic condition on M). Then we compute

$$r_1 := \begin{pmatrix} \underline{v} \\ \underline{c} \\ 0 \end{pmatrix}\,, \quad r_2 := \begin{pmatrix} \dfrac{1+M^2}{1-M^2}\,\underline{v} \\ \underline{c} \\ \dfrac{2\,\underline{u}}{1-M^2} \end{pmatrix}\,, \quad r_3 := \begin{pmatrix} 0 \\ \underline{c} \\ \underline{u} \end{pmatrix}\,,$$

and

$$\begin{aligned} &\ell_1 := \begin{pmatrix} \dfrac{1}{2\,\underline{v}\,\underline{u}} & \dfrac{1}{2\,\underline{u}\,\underline{c}} & 0 \end{pmatrix}\,, \quad \ell_2 := -\frac{1}{1+M^2} \begin{pmatrix} \dfrac{1+M^2}{2\,\underline{v}\,\underline{u}} & \dfrac{1-M^2}{2\,\underline{u}\,\underline{c}} & \dfrac{1}{\underline{c}^2} \end{pmatrix}\,, \\ &\ell_3 := \frac{1}{1+M^2} \begin{pmatrix} 0 & \dfrac{1}{\underline{u}\,\underline{c}} & \dfrac{1}{\underline{c}^2} \end{pmatrix}\,. \end{aligned}$$

We now make the choice of B precise. As in [CGW14, Appendix B], we choose

$$B := \begin{pmatrix} 0 & \underline{v} & 0 \\ \underline{u} & 0 & \underline{v} \end{pmatrix}\,,$$

which does not have any physical interpretation but makes the following calculations rather easy. The reader can check that Assumptions 1.3 and 1.6 are satisfied, and we can choose $e := r_1 - r_3$ as the vector that spans $\ker B \cap \mathbb{E}^s(\underline{\tau}, \underline{\eta})$. The one-dimensional space $B\,\mathbb{E}^s(\underline{\tau}, \underline{\eta})$ can be written as the orthogonal of the kernel of the linear form $\underline{b} := (\underline{u}, -\underline{c})$. We can then compute the bilinear Fourier multiplier Q_{per} defined in (2.20), and get:

$$Q_{\text{per}}[a, \widetilde{a}] = \underline{v}\,\underline{u}\,\underline{c} \sum_{k \in \mathbb{Z}} \left(\sum_{\substack{k_1 + k_3 = k, \\ k_1\,k_3 \ne 0}} \left(1 + \frac{2\,(1-M^2)\,k_1}{k_3 + (2\,k_1 + k_3)\,M^2} \right) a_{k_1}\,\widetilde{a}_{k_3} \right) \mathrm{e}^{2\,i\,\pi\,k\,\theta_0/\Theta}\,,$$

whose kernel is unbounded and depends, as expected, on the arithmetic properties of M^2.

To conclude this Appendix, let us observe that in three space dimensions, the isentropic Euler equations are no longer strictly hyperbolic but they enjoy a conservative structure (in the physical variables $\rho, \rho\,\vec{u}$). Similarly, the full Euler equations with the energy conservation law also have a conservative structure. Hence Assumption 1.2 is satisfied, and one can perform a similar derivation as above for the leading amplitude equation.

B Formal derivation of the large period limit: from wavetrains to pulses

B.1 The large period limit of the amplitude equation (2.19)

In this Appendix, we study the relationship between the quadratic operators arising in the leading amplitude equations for the wavetrains and pulses problems which we have considered. For the sake of clarity, we focus on the easiest case $N=3$, $p=2$, that was considered in paragraph 2.2 and section 5. The Fourier multiplier Q_{per} is then defined by (2.20), while the bilinear operator Q_{pul} is defined by (5.31). From these expressions, we can decompose both operators as

$$Q_{\text{per}}[a,a] = (\underline{b}\,B\,r_2)\,\ell_2\,E_{1,3}\,\widetilde{\mathbb{F}}_{\text{per}}[\partial_\theta a,a] + (\underline{b}\,B\,r_2)\,\ell_2\,E_{3,1}\,\widetilde{\mathbb{F}}_{\text{per}}[a,\partial_\theta a]\,,$$
$$Q_{\text{pul}}[a,a] = (\underline{b}\,B\,r_2)\,\ell_2\,E_{1,3}\,\widetilde{\mathbb{F}}_{\text{pul}}[\partial_\theta a,a] + (\underline{b}\,B\,r_2)\,\ell_2\,E_{3,1}\,\widetilde{\mathbb{F}}_{\text{pul}}[a,\partial_\theta a]\,,$$

with (observe the slightly different normalizations with respect to (3.3) and (6.2)):

$$\text{(B.1)} \qquad \widetilde{\mathbb{F}}_{\text{per}}[u,v](\theta) := \frac{i\,\Theta}{2\,\pi}\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{k_1+k_3=k,\\ k_1\,k_3\neq 0}} \frac{u_{k_1}\,v_{k_3}}{k_1\,(\underline{\omega}_1-\underline{\omega}_2)+k_3\,(\underline{\omega}_3-\underline{\omega}_2)}\right)\mathrm{e}^{2\,i\,\pi\,k\,\theta/\Theta}\,,$$

$$\text{(B.2)} \qquad \widetilde{\mathbb{F}}_{\text{pul}}[u,v](\theta) := \int_0^{+\infty} u(\theta+(\underline{\omega}_1-\underline{\omega}_2)\,s)\,v(\theta+(\underline{\omega}_3-\underline{\omega}_2)\,s)\,\mathrm{d}s\,.$$

In (B.1), both functions u,v are assumed to be Θ-periodic and u_k, v_k stand for their k-th Fourier coefficient, while in (B.2), u,v are defined on $\mathbb{R}$ and have sufficiently fast decay at infinity (so that the integral makes sense).

Our goal is to explain how one can compute the (formal) limit of $\widetilde{\mathbb{F}}_{\text{per}}$ when the period Θ becomes infinitely large and to make the link with the expression of $\widetilde{\mathbb{F}}_{\text{pul}}$ in (B.2). The additional variables (t,y) play the role of parameters here, so we focus on $\widetilde{\mathbb{F}}_{\text{per}}, \widetilde{\mathbb{F}}_{\text{pul}}$ as operators acting on functions that depend on a single variable θ. We pick two functions u,v in the Schwartz class $\mathcal{S}(\mathbb{R})$, and define

$$\forall\,\theta\in\mathbb{R}\,,\quad u_\Theta(\theta) := \sum_{n\in\mathbb{Z}} u(\theta+n\,\Theta)\,,\quad v_\Theta(\theta) := \sum_{n\in\mathbb{Z}} v(\theta+n\,\Theta)\,.$$

The functions u_Θ, v_Θ are Θ-periodic and converge, uniformly on compact sets, towards u,v as Θ tends to infinity. Moreover, the Poisson summation formula gives the Fourier coefficients of u_Θ, v_Θ in terms of the Fourier transform of u,v:

$$u_\Theta(\theta) = \frac{1}{\Theta}\sum_{k\in\mathbb{Z}}\widehat{u}\left(\frac{2\,k\,\pi}{\Theta}\right)\mathrm{e}^{2\,i\,\pi\,k\,\theta/\Theta}\,,\quad v_\Theta(\theta) = \frac{1}{\Theta}\sum_{k\in\mathbb{Z}}\widehat{v}\left(\frac{2\,k\,\pi}{\Theta}\right)\mathrm{e}^{2\,i\,\pi\,k\,\theta/\Theta}\,.$$

We compute

$$\begin{aligned}\widetilde{\mathbb{F}}_{\text{per}}[u_\Theta,v_\Theta](\theta) &= \frac{i}{2\,\pi\,\Theta}\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{k_1+k_3=k,\\ k_1\,k_3\neq 0}}\frac{\widehat{u}(2\,k_1\,\pi/\Theta)\,\widehat{v}(2\,k_3\,\pi/\Theta)}{k_1\,(\underline{\omega}_1-\underline{\omega}_2)+k_3\,(\underline{\omega}_3-\underline{\omega}_2)}\right)\mathrm{e}^{2\,i\,\pi\,k\,\theta/\Theta}\\ &= \frac{i}{4\,\pi^2}\,\frac{4\,\pi^2}{\Theta^2}\sum_{k\in\mathbb{Z}}\sum_{\substack{k_1+k_3=k,\\ k_1\,k_3\neq 0}}\frac{\widehat{u}(2\,k_1\,\pi/\Theta)\,\widehat{v}(2\,k_3\,\pi/\Theta)}{(2\,k_1\,\pi/\Theta)\,(\underline{\omega}_1-\underline{\omega}_2)+(2\,k_3\,\pi/\Theta)\,(\underline{\omega}_3-\underline{\omega}_2)}\,\mathrm{e}^{2\,i\,\pi\,k\,\theta/\Theta}\,.\end{aligned}$$

The latter expression suggests that $\widetilde{\mathbb{F}}_{\mathrm{per}}[u_\Theta, v_\Theta](\theta)$ is the approximation by a Riemann sum, of the double integral[14]

$$\frac{i}{4\,\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\widehat{u}(\eta)\,\widehat{v}(\xi-\eta)}{\eta\,(\underline{\omega}_1-\underline{\omega}_2)+(\xi-\eta)\,(\underline{\omega}_3-\underline{\omega}_2)}\,\mathrm{e}^{i\,\xi\,\theta}\,\mathrm{d}\eta\,\mathrm{d}\xi\,.$$

Formally, this means that, as Θ tends to infinity, $\widetilde{\mathbb{F}}_{\mathrm{per}}[u_\Theta, v_\Theta]$ tends towards a function whose Fourier transform is given by

$$\xi\in\mathbb{R}\longmapsto-\frac{1}{2\,i\,\pi}\int_{\mathbb{R}}\frac{\widehat{u}(\eta)\,\widehat{v}(\xi-\eta)}{\eta\,(\underline{\omega}_1-\underline{\omega}_2)+(\xi-\eta)\,(\underline{\omega}_3-\underline{\omega}_2)}\,\mathrm{d}\eta\,,\tag{B.3}$$

assuming of course that the latter expression makes any sense. We wish to compare (B.3) with the Fourier transform of $\mathbb{F}_{\mathrm{pul}}[u,v]$, whose expression is given by the following (rigorous!) result.

Lemma B.1. *Let $u, v\in\mathcal{S}(\mathbb{R})$. Then (B.2) defines a function $\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v]\in\mathcal{S}(\mathbb{R})$ whose Fourier transform is given by*

$$\begin{aligned}\xi\in\mathbb{R}\longmapsto\frac{1}{2\,|\underline{\omega}_3-\underline{\omega}_1|}\,\widehat{u}\left(\frac{\underline{\omega}_3-\underline{\omega}_2}{\underline{\omega}_3-\underline{\omega}_1}\,\xi\right)\,\widehat{v}\left(\frac{\underline{\omega}_2-\underline{\omega}_1}{\underline{\omega}_3-\underline{\omega}_1}\,\xi\right)&\\ -\frac{1}{2\,i\,\pi}\,\mathrm{p.v.}\int_{\mathbb{R}}\frac{\widehat{u}(\eta)\,\widehat{v}(\xi-\eta)}{\eta\,(\underline{\omega}_1-\underline{\omega}_2)+(\xi-\eta)\,(\underline{\omega}_3-\underline{\omega}_2)}\,\mathrm{d}\eta\,,&\end{aligned}\tag{B.4}$$

where p.v. *stands for the principal value of the integral.*

Proof. That $\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v]$ belongs to $\mathcal{S}(\mathbb{R})$ is done in two steps. One first proves differentiability by applying the classical differentiation Theorem for integrals with a parameter. Then the formula

$$\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v]'=\widetilde{\mathbb{F}}_{\mathrm{pul}}[u',v]+\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v']\,,$$

and a straightforward induction argument yields infinite differentiability. Given an integer N, we can apply the Peetre inequality and get

$$(1+\theta^2)^N\,|\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v](\theta)|\le C\,\sup_{t\in\mathbb{R}}\left((1+t^2)^N\,|v(t)|\right)\int_0^{+\infty}(1+(\theta+(\underline{\omega}_1-\underline{\omega}_2)\,s)^2)^N\,|u(\theta+(\underline{\omega}_1-\underline{\omega}_2)\,s)|\,\mathrm{d}s\,,$$

so $(1+\theta^2)^N\,\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v]$ is bounded. The previous formula for $\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v]'$ shows again by induction that all functions $(1+\theta^2)^{N_1}\,\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v]^{(N_2)}$ are bounded. We may thus compute the Fourier transform of $\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v]$.

Let us first write

$$\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v](\theta)=F_1(\theta)+F_2(\theta)\,,\tag{B.5}$$

with

$$\begin{aligned}F_1(\theta)&:=\frac{1}{2}\int_{\mathbb{R}}u(\theta+(\underline{\omega}_1-\underline{\omega}_2)\,s)\,v(\theta+(\underline{\omega}_3-\underline{\omega}_2)\,s)\,\mathrm{d}s\,,\\ F_2(\theta)&:=\frac{1}{2}\int_{\mathbb{R}}\mathrm{sgn}(s)\,u(\theta+(\underline{\omega}_1-\underline{\omega}_2)\,s)\,v(\theta+(\underline{\omega}_3-\underline{\omega}_2)\,s)\,\mathrm{d}s\,.\end{aligned}$$

[14]It is not obvious at first sight that this double integral makes any sense, but our goal here is to identify formally the large period limit so let us pretend that all manipulations on the integrals and limits are valid.

Here sgn denotes the sign function. The Fourier transform of F_1 is computed by applying an elementary change of variables and the Fubini Theorem:

$$
\begin{aligned}
\widehat{F_1}(\xi) &= \frac{1}{2}\int_{\mathbb{R}^2} \mathrm{e}^{-i\,\xi\,\theta}\, u(\theta + (\underline{\omega}_1 - \underline{\omega}_2)\, s)\, v(\theta + (\underline{\omega}_3 - \underline{\omega}_2)\, s)\, \mathrm{d}s\, \mathrm{d}\theta \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \mathrm{e}^{-i\,\xi\,\dfrac{X\,(\underline{\omega}_3-\underline{\omega}_2) - Y\,(\underline{\omega}_1-\underline{\omega}_2)}{\underline{\omega}_3-\underline{\omega}_1}}\, u(X)\, v(Y)\, \frac{\mathrm{d}X\, \mathrm{d}Y}{|\underline{\omega}_3 - \underline{\omega}_1|}\,.
\end{aligned}
$$

The contribution of $\widehat{F_1}(\xi)$ therefore gives the term in the first line of (B.4). It remains to compute the Fourier transform of F_2.

The Fubini Theorem gives

$$
\widehat{F_2}(\xi) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{sgn}(s) \left(\int_{\mathbb{R}} \mathrm{e}^{-i\,\xi\,\theta}\, u(\theta + (\underline{\omega}_1 - \underline{\omega}_2)\, s)\, v(\theta + (\underline{\omega}_3 - \underline{\omega}_2)\, s)\, \mathrm{d}\theta\right) \mathrm{d}s = \frac{1}{4\,\pi}\int_{\mathbb{R}} \mathrm{sgn}(s)\, (\widehat{U_s}\star\widehat{V_s})(\xi)\, \mathrm{d}s,
$$

with

$$
U_s(\theta) := u(\theta + (\underline{\omega}_1 - \underline{\omega}_2)\, s)\,, \quad V_s(\theta) := v(\theta + (\underline{\omega}_3 - \underline{\omega}_2)\, s)\,.
$$

We thus get

$$
\widehat{F_2}(\xi) = \frac{1}{4\,\pi}\int_{\mathbb{R}} \mathrm{sgn}(s) \left(\mathrm{e}^{i\,(\underline{\omega}_1-\underline{\omega}_2)\,s\,\xi} \int_{\mathbb{R}} \mathrm{e}^{i\,(\underline{\omega}_3-\underline{\omega}_1)\,s\,\eta}\, \widehat{u}(\xi-\eta)\, \widehat{v}(\eta)\, \mathrm{d}\eta\right) \mathrm{d}s\,,
$$

which is an expression of the form

$$
\frac{1}{4\,\pi}\int_{\mathbb{R}} \mathrm{sgn}(s)\, \widehat{S_\xi}(s)\, \mathrm{d}s\,,
$$

for a suitable Schwartz function S_ξ (ξ is a parameter here). We can therefore transform the expression of $\widehat{F_2}(\xi)$ by using the Fourier transform of the sign function, which yields

$$
\widehat{F_2}(\xi) = \frac{1}{2\,i\,\pi}\, \text{p.v.}\int_{\mathbb{R}} \frac{1}{\eta}\, S_\xi(\eta)\, \mathrm{d}\eta\,.
$$

The function S_ξ is given by

$$
S_\xi(\eta) = \frac{1}{|\underline{\omega}_3 - \underline{\omega}_1|}\, \widehat{u}\left(\frac{\eta + (\underline{\omega}_3-\underline{\omega}_2)\,\xi}{\underline{\omega}_3-\underline{\omega}_1}\right) \widehat{v}\left(\frac{\eta + (\underline{\omega}_1-\underline{\omega}_2)\,\xi}{\underline{\omega}_1-\underline{\omega}_3}\right)\,,
$$

and a last change of variable gives, as claimed in (B.4):

$$
\widehat{F_2}(\xi) = -\frac{1}{2\,i\,\pi}\, \text{p.v.}\int_{\mathbb{R}} \frac{\widehat{u}(\eta)\, \widehat{v}(\xi-\eta)}{\eta\,(\underline{\omega}_1-\underline{\omega}_2) + (\xi-\eta)\,(\underline{\omega}_3-\underline{\omega}_2)}\, \mathrm{d}\eta\,.
$$

$\square$

In view of the decomposition (B.5), and the previous computations of Fourier transforms, we observe that the formal limit (B.3) for the Fourier transform of $\widetilde{\mathbb{F}}_{\mathrm{per}}[u_\Theta, v_\Theta]$ does not coincide with the Fourier transform of $\widetilde{\mathbb{F}}_{\mathrm{pul}}[u,v]$ but rather coincides with the Fourier transform of the function F_2. In other words, we have *formally* obtained that in the large period limit $\Theta \to +\infty$, $\widetilde{\mathbb{F}}_{\mathrm{per}}[u_\Theta, v_\Theta]$ tends towards

$$
\text{(B.6)} \qquad \frac{1}{2}\int_{\mathbb{R}} \mathrm{sgn}(s)\, u(\theta + (\underline{\omega}_1 - \underline{\omega}_2)\, s)\, v(\theta + (\underline{\omega}_3 - \underline{\omega}_2)\, s)\, \mathrm{d}s\,,
$$

which is a "symmetrized" version of the operator $\widetilde{\mathbb{F}}_{\text{pul}}$ in (B.2). In particular, for any function $a \in \mathcal{S}(\mathbb{R})$, the Θ-periodic function $\widetilde{\mathbb{F}}_{\text{per}}[\partial_\theta a_\Theta, a_\Theta]$ formally converges, as Θ tends to infinity, towards

$$\frac{1}{2} \int_{\mathbb{R}} \text{sgn}(s)\, \partial_\theta a(\theta + (\underline{\omega}_1 - \underline{\omega}_2)\, s)\, a(\theta + (\underline{\omega}_3 - \underline{\omega}_2)\, s)\, \mathrm{d}s\,. \tag{B.7}$$

Remark B.2. *It might be surprising that the small divisors condition in Assumption 1.9 does not seem to play any role in the analysis of pulses. However, there remains some kind of trace of this Assumption but it is hidden in the functional framework. More precisely, we have shown that for some large enough index ν, the bilinear operator $\mathbb{F}_{\text{pul}}$ satisfies a bound of the form*

$$\|\mathbb{F}_{\text{pul}}(\partial_\theta u, \partial_\theta v)\|_{\Gamma^\nu} \le C\, \|u\|_{\Gamma^\nu}\, \|v\|_{\Gamma^\nu}\,,$$

where Γ^ν is a weighted Sobolev space. Omitting the variables (t,y) for simplicity, it is also possible to prove that for all integer ν, there exists a bounded sequence $(a_n)_{n\in\mathbb{N}}$ in $H^\nu(\mathbb{R})$ such that

$$\langle a_n, \mathbb{F}_{\text{pul}}(\partial_\theta a_n, \partial_\theta a_n)\rangle_{H^\nu(\mathbb{R})} \to +\infty\,.$$

In other words, the space Γ^ν is well-suited for studying boundedness of $\mathbb{F}_{\text{pul}}$ and the standard Sobolev space H^ν is not. This is not so surprising because on the Fourier side, the kernel of $\mathbb{F}_{\text{pul}}$ is unbounded, which is a trace of the small divisors in the wavetrains problem. Unboundedness of the kernel requires introducing a principal value in Lemma B.1, which relies on some continuity of the integrand. It is therefore not surprising that continuity of $\mathbb{F}_{\text{pul}}$ holds in a functional space where Fourier transforms have some extra regularity properties (which is another way to formulate polynomial decay in the physical variable).

As repeatedly claimed, this paragraph does not aim at giving a *rigorous* justification of the large period limit. An open question that is raised by the formal arguments given above is: assuming that the small divisors condition in Assumption 1.9 holds, proving that the Θ-periodic function $\widetilde{\mathbb{F}}_{\text{per}}[u_\Theta, v_\Theta]$ does indeed converge (for instance, uniformly on compact sets), as Θ tends to infinity, towards (B.6) for u, v in the Schwartz class. It is also likely that the amplitude equation (6.1) is ill-posed in $H^\nu(\mathbb{R}^d)$ for any large integer ν, though it is well-posed in Γ^ν. We refer to [BGCT11] for similar ill-posedness issues on nonlocal versions of the Burgers equation.

B.2 What is the correct amplitude equation for Mach stem formation ?

It is surprising that the formal limit of the amplitude equation (2.19) as Θ tends to infinity does not coincide with (5.30) if the operator Q_{pul} is defined by (5.31). In several problems of nonlinear geometric optics, namely

- Quasilinear hyperbolic Cauchy problems [HMR86, JMR95, AR02, GR06],
- Uniformly stable quasilinear hyperbolic boundary value problems [Wil99, Wil02, CGW11, CW13],
- Weakly stable semilinear hyperbolic boundary value problems [CGW14, CW14],

the limit of the amplitude equation for wavetrains does coincide with the amplitude equation for pulses. Of course, nonlinear geometric optics has received a more complete description for these three problems than for the one we consider here because in all three above problems, *exact* solutions on a *fixed* time interval have been shown to exist and to be close to *approximate* WKB solutions (which justifies the relevance of the corresponding evolution equation for the leading order amplitudes).

In view of all available references in the literature, it is therefore natural to wonder whether the amplitude equation (5.30), (5.31), which is the one derived in [MR83], does give an accurate description for pulse-like solutions to (1.1). Rephrasing the question, is it possible to construct another family of approximate solutions that would be determined by solving an amplitude equation that is obtained as the large period limit of (2.19), (2.20) ? The answer is yes, but the price to pay is to forget about boundedness of the second corrector $\mathcal{U}_2$.

Let us quickly review the analysis of the WKB cascade (5.2), (5.4) for pulses. In Step 1 of Paragraph 5.2, we have derived the expression (5.21) of the leading order profile $\mathcal{U}_0$ by assuming that the *first* corrector $\mathcal{U}_1$ is bounded. The amplitudes σ_1, σ_3 solve the decoupled Burgers equations (5.19) and satisfy the trace relation (5.20). In Step 4 of Paragraph 5.2, we have obtained the expression of the component of $E_P \mathcal{U}_1$ on r_2 by assuming that the second corrector $\mathcal{U}_2$ is bounded. However, this boundedness assumption is not necessarily supported by a clear physical interpretation, and it might very well be that $\mathcal{U}_2$ is not uniformly bounded in the fast variables (θ_0, ξ_d).

One can indeed construct (infinitely) many families of approximate solutions to (1.1). Let us choose a parameter $s \in [0,1]$. Then, given the leading order profile $\mathcal{U}_0$ in (5.21) with functions σ_1, σ_3 that are expected to decay sufficiently fast at infinity, one can construct the first corrector $\mathcal{U}_1$ as a particular solution to (5.2)(b). One possible choice is:

$$\widetilde{\mathcal{U}}_{1,s} := \sum_{m=1}^{3} \left\{ -s \int_{\xi_d}^{+\infty} \ell_m \, \mathcal{F}_0(t,x,\theta_0 + \underline{\omega}_m \, (\xi_d - X), X) \, \mathrm{d}X \right. \\ \left. + (1-s) \int_{-\infty}^{\xi_d} \ell_m \, \mathcal{F}_0(t,x,\theta_0 + \underline{\omega}_m \, (\xi_d - X), X) \, \mathrm{d}X \right\} r_m \, ,$$

and the general solution to (5.2)(b) is of the form:

$$\widetilde{\mathcal{U}}_{1,s} + \sum_{m=1}^{3} \tau_m^s(t,x,\theta_0 + \underline{\omega}_m \, \xi_d) \, r_m \, .$$

The choice in [MR83] corresponds, as in Paragraph 5.2, to $s = 1$ and causality is invoked to set the function τ_2^1 equal to zero (see Equation (4.10) in [MR83]). In Paragraph 5.2, we have explained why $\tau_2^1 = 0$ can de deduced from the assumption that there exists a bounded corrector to (5.2)(c).

If we do not assume existence of a bounded corrector to (5.2)(c), there does not seem to be a clear option for constructing the outgoing part τ_2^s, and therefore, given $s \in [0,1]$, the corrector we have at our disposal satisfies

$$\widetilde{\mathcal{U}}_{1,s}(t,y,0,\theta_0,0) = \star \, r_1 - s \int_0^{+\infty} \ell_2 \, \mathcal{F}_0(t,y,0,\theta_0 - \underline{\omega}_2 \, X, X) \, \mathrm{d}X \, r_2 \\ + (1-s) \int_{-\infty}^{0} \ell_2 \, \mathcal{F}_0(t,y,0,\theta_0 - \underline{\omega}_2 \, X, X) \, \mathrm{d}X \, r_2 + \star \, r_3 \, .$$

Plugging the latter expression in (5.4)(b), applying the row vector $\underline{b}$ and differentiating with respect to

θ_0, we obtain Equation (5.30) with the following new definition of the operator $Q_{\rm pul}$:

$$
\begin{aligned}
Q_{\rm pul}[a,\widetilde{a}](\theta_0) := (\underline{b}\, B\, r_2)\, \ell_2\, E_{1,3}\, s \int_0^{+\infty} \partial_{\theta_0} a(\theta_0 + (\underline{\omega}_1 - \underline{\omega}_2)\, X)\, \widetilde{a}(\theta_0 + (\underline{\omega}_3 - \underline{\omega}_2)\, X)\, {\rm d}X \\
+ (\underline{b}\, B\, r_2)\, \ell_2\, E_{3,1}\, s \int_0^{+\infty} a(\theta_0 + (\underline{\omega}_1 - \underline{\omega}_2)\, X)\, \partial_{\theta_0} \widetilde{a}(\theta_0 + (\underline{\omega}_3 - \underline{\omega}_2)\, X)\, {\rm d}X \\
- (\underline{b}\, B\, r_2)\, \ell_2\, E_{1,3}\, (1-s) \int_{-\infty}^0 \partial_{\theta_0} a(\theta_0 + (\underline{\omega}_1 - \underline{\omega}_2)\, X)\, \widetilde{a}(\theta_0 + (\underline{\omega}_3 - \underline{\omega}_2)\, X)\, {\rm d}X \\
- (\underline{b}\, B\, r_2)\, \ell_2\, E_{3,1}\, (1-s) \int_{-\infty}^0 a(\theta_0 + (\underline{\omega}_1 - \underline{\omega}_2)\, X)\, \partial_{\theta_0} \widetilde{a}(\theta_0 + (\underline{\omega}_3 - \underline{\omega}_2)\, X)\, {\rm d}X\, .
\end{aligned}
\tag{B.8}
$$

Unlike all other terms in (5.30), the bilinear term $Q_{\rm pul}$ does depend on the value of s that is chosen for constructing the particular solution $\widetilde{\mathcal{U}}_{1,s}$ to (5.2)(b), and therefore invoking causality to discard the τ_2 term has an impact on the leading order amplitude equation (5.30). The choice $s = 1/2$ is the only that is compatible with the large period limit $\Theta \to +\infty$.

Our overall assessment is the following. In the wavetrain problem, the construction of approximate WKB solutions in Part I is well-understood. In particular Theorem 1.10 shows that the full WKB cascade has a *unique* solution $(\mathcal{U}_n)_{n \in \mathbb{N}}$ in a suitable functional space. There remains of course the problem of understanding the lifespan of exact solutions to (1.1) and whether approximate and exact solutions are close to each other. In the pulse problem, the situation is worse since the construction of approximate solutions in Part II may be questionable. There are basically two options (the infinitely many other ones seem to be a mathematical artifact). Either one hopes that the expansion of exact solutions will yield a second corrector $\mathcal{U}_2$ that is bounded, and in that case the appropriate leading amplitude equation is given, as in [MR83], by (5.30), (5.31). Or one rather expects that the amplitude equation for pulses should coincide with the large period limit of the corresponding wavetrain equation, and in that case the appropriate leading amplitude equation is still given by (5.30) but with the bilinear operator $Q_{\rm pul}$ as in (B.8) with $s = 1/2$. The two corresponding leading profiles differ in each of the two options. Without any precise understanding of the behavior of exact solutions, deciding between the two possible expansions seems hardly possible.

C Some remarks on the resonant case

In this Appendix, we explain why the analysis in Part I breaks down when resonances occur. Let us consider for simplicity the 3×3 strictly hyperbolic case of Paragraph 2.2. A resonance corresponds to the existence of a triplet $(n_1, n_2, n_3) \in \mathbb{Z}^3$, with $n_1\, n_2\, n_3 \neq 0$, gcd $(n_1, n_2, n_3) = 1$, and

$$
n_1\, \varphi_1 = n_2\, \varphi_2 + n_3\, \varphi_3\, . \tag{C.1}
$$

Incoming waves associated with the phases φ_1, φ_3 may then interact to produce an outgoing wave for the phase φ_2. More precisely, the source term $\varepsilon^2\, G(t, y, \varphi_0/\varepsilon)$ in (1.1) is expected to give rise to incoming waves associated with the phases φ_1, φ_3 of amplitude $O(\varepsilon)$. Then the products $A_j(u_\varepsilon)\, \partial_j u_\varepsilon$ in (1.1) ignites oscillations associated with the outgoing phase φ_2 of amplitude[15] $O(\varepsilon)$. These φ_2-oscillations are then

[15]This is a major scaling difference with our previous work [CGW14] where nonlinear interaction was due to a zero order term so the outgoing oscillations produced by resonances had amplitude $O(\varepsilon^2)$, instead of $O(\varepsilon)$ here.

expected to be amplified when reflected on the boundary, as in the linear analysis of [CG10], but this would give rise to φ_1, φ_3-oscillations of amplitude $O(1)$ and would thus completely ruin the ansatz (2.5). Resonances are therefore expected to produce dramatically different phenomena from the ones that are studied here.

Another hint that resonances should make the ansatz (2.5) break down is to try to solve the WKB cascade (4.1), (4.3) when the resonance (C.1) occurs between two incoming phases φ_1, φ_3 and one outgoing phase φ_2. Let us recall that in the 3×3 strictly hyperbolic case considered in Paragraph 2.2, B is a 2×3 matrix of maximal rank. Its kernel has dimension 1 and is therefore spanned by the vector $e = e_1 + e_3 \in \mathbb{E}^s(\underline{\tau}, \underline{\eta})$. In other words, we can choose a vector $\check{e} \in \mathbb{E}^s(\underline{\tau}, \underline{\eta})$ such that

$$\mathbb{R}^2 = \text{Span}\,(B\,\check{e}, B\,r_2)\,.$$

We now follow the analysis of Paragraph 2.2 and try to analyze the WKB cascade in the presence of a resonance. Equation (2.6)(a) shows again that the leading profile can be decomposed as

$$\mathcal{U}_0(t,x,\theta_1,\theta_2,\theta_3) = \underline{\mathcal{U}}_0(t,x) + \sum_{m=1}^{3} \sigma_m(t,x,\theta_m)\, r_m\,.$$

Then Equation (2.6)(b) shows that the mean value $\underline{\mathcal{U}}_0$ satisfies (2.10), (2.11), and therefore vanishes. The interior equation satisfied by each σ_m exhibits the resonance between the phases, see [Rau12]. Namely, the σ_m's satisfy the coupled Burgers-type equations

$$\text{(C.2)} \qquad \begin{cases} \partial_t \sigma_1 + \mathbf{v}_1 \cdot \nabla_x \sigma_1 + c_1\, \sigma_1\, \partial_{\theta_1} \sigma_1 = B_1(\sigma_2,\sigma_3)\,, \\ \partial_t \sigma_2 + \mathbf{v}_2 \cdot \nabla_x \sigma_2 + c_2\, \sigma_2\, \partial_{\theta_2} \sigma_2 = B_2(\sigma_1,\sigma_3)\,, \\ \partial_t \sigma_3 + \mathbf{v}_3 \cdot \nabla_x \sigma_3 + c_3\, \sigma_3\, \partial_{\theta_3} \sigma_3 = B_3(\sigma_1,\sigma_2)\,, \end{cases}$$

where the constants c_m are defined as in (2.12) and, for instance:

$$\begin{aligned} B_1(\sigma_2,\sigma_3)(t,x,\theta_1) := &\frac{2\,i\,\pi\,\ell_1\,\partial_j\varphi_3\,(\mathrm{d}A_j(0)\cdot r_2)\,r_3}{\Theta\,\ell_1\,r_1} \sum_{k\in\mathbb{Z}^*} c_{k\,n_2}(\sigma_2)(t,x)\,(k\,n_3)\,c_{k\,n_3}(\sigma_3)(t,x)\,\mathrm{e}^{2\,i\,\pi\,k\,n_1\,\theta_1/\Theta} \\ &+ \frac{2\,i\,\pi\,\ell_1\,\partial_j\varphi_2\,(\mathrm{d}A_j(0)\cdot r_3)\,r_2}{\Theta\,\ell_1\,r_1} \sum_{k\in\mathbb{Z}^*} (k\,n_2)\,c_{k\,n_2}(\sigma_2)(t,x)\,c_{k\,n_3}(\sigma_3)(t,x)\,\mathrm{e}^{2\,i\,\pi\,k\,n_1\,\theta_1/\Theta}\,. \end{aligned}$$

The definitions of $B_2(\sigma_1,\sigma_3)$ and $B_3(\sigma_1,\sigma_2)$ are similar, and correspond to the so-called *interaction integrals* in [CGW11].

The boundary conditions for the σ_m's are given by (2.8)(a). Using the basis $(e,\check{e})$ of Span (r_1,r_3), we decompose

$$\mathcal{U}_0(t,y,0,\theta_0,\theta_0,\theta_0) = a(t,y,\theta_0)\,e + \check{a}(t,y,\theta_0)\,\check{e} + \sigma_2(t,y,0,\theta_0)\,r_2\,,$$

so (2.8)(a) gives

$$\check{a} \equiv 0\,, \quad \sigma_2|_{x_d=0} \equiv 0\,.$$

However, this boundary condition on σ_2 does not seem to be compatible with (C.2) because σ_2 satisfies an outgoing transport equation with a nonzero source term (it is proved in [CGW11] that the operators B_1, B_2, B_3 in (C.2) act like *semilinear* terms and do not contribute to the leading order part of the differential operators in (C.2)). Even if we manage to isolate an amplitude equation for determining the traces of the incoming amplitudes σ_1, σ_3, the overall cascade seems to give rise to an overdetermined problem for $\mathcal{U}_0$.

We do not claim of course that the above arguments give a rigorous justification of the ill-posedness of (4.1), (4.3) when a resonance occurs, but it clearly indicates that the ansatz (2.5) does not seem to be appropriate anylonger.

When there is only one incoming phase, all the above discussion becomes irrelevant, and this may again be one reason why any discussion on resonances is absent from [AM87] or [WY13, WY14].

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